

ON MODEL-THEORETIC CONNECTED COMPONENTS IN SOME GROUP EXTENSIONS

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ABSTRACT. Assume G is a group acting by automorphisms on an abelian group A and that all the data is definable in a first order structure \mathcal{G} . Suppose $h: G \times G \rightarrow A$ is a B -definable (in \mathcal{G}) 2-cocycle with finite image contained in B (for some finite set B) and \tilde{G} is the corresponding extension of G by A . Let \mathcal{G}^* be a monster model of $\text{Th}(\mathcal{G})$ and \tilde{G}^* the interpretation of \tilde{G} in \mathcal{G}^* . We prove that under some general hypothesis, $\tilde{G}_{B}^{*000} \neq \tilde{G}_{B}^{*00}$, where \tilde{G}_{B}^{*00} is the smallest B -type-definable in \mathcal{G}^* subgroup of \tilde{G}^* of bounded index and \tilde{G}_{B}^{*000} is the smallest invariant under $\text{Aut}(\mathcal{G}^*/B)$ subgroup of \tilde{G}^* of bounded index.

We apply this theorem to produce new classes of examples of groups for which the smallest B -type-definable subgroup of bounded index differs from the smallest B -invariant subgroup of bounded index.

1. INTRODUCTION

Assume G is a group \emptyset -definable in a monster model of some first order theory, and let B be a (small) set of parameters from this model. The following connected components play a very important role in the study of groups from the model-theoretic perspective:

- the intersection of all B -definable subgroups of G of finite index, denoted by G_B^0 ,
- the smallest B -type-definable subgroup of G of bounded index, denoted by G_B^{00} ,
- the smallest B -invariant subgroup of G of bounded index, denoted by G_B^{000} .

It is clear that $G_B^{000} \leq G_B^{00} \leq G_B^0 \leq G$, and it is easy to show that all these groups are normal in G . Sometimes these connected components do not depend on the choice of B , e.g. in NIP theories. In such a situation, we skip the parameter set B , and we say that the appropriate connected component exists, e.g. we write G^{000} and say that G^{000} exists (notice that then G^{000} is the smallest invariant, over a small set of parameters, subgroup of bounded index).

The significance of the above connected components was discussed in various papers (e.g. see [7, 6]). We will say only a few words about the motivation. Originally, G^0 played a fundamental role in the study of generic types in stable groups, for example due to the fact that G/G^0 is always a profinite group. The importance of G^{00} stems from the fact that it allows to associate with G a compact topological group G/G^{00} (with the so-called logic topology). This becomes particularly interesting in ω -minimal structures due to

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Pillay's conjecture which describes G/G^{00} as a compact Lie group of an appropriate dimension, and so associates with the group G a classical mathematical object G/G^{00} [12]. A common motivation to consider all three connected components also comes from their relationships with strong types in various senses (strong types, Kim-Pillay types and Lascar strong types), which in turn are essential notions in the study of stable, simple and NIP theories. For details on these relationships see [7, Section 3].

Our main goal is to find examples or, more desirably, general methods of constructing them, of groups G for which $G_B^{00} \neq G_B^{000}$. This problem arose during the work on the first author's Ph.D. thesis under the supervision of L. Newelski, and it appears explicitly for example in [7]. Recall that any such example leads to a new G -compact theory (see [7, Corollary 3.6]).

In [3], the authors have found first (strongly related to each other) examples of groups G for which $\widetilde{G}^{00} \neq \widetilde{G}^{000}$. Their example is a monster model $\widetilde{\text{SL}}_2(\mathbb{R})$ of the topological universal cover $\text{SL}_2(\mathbb{R})$ of $\text{SL}_2(\mathbb{R})$. The proof uses the fact that $\text{SL}_2(\mathbb{R})$ is a central extension of $\text{SL}_2(\mathbb{R})$ by \mathbb{Z} given by a definable 2-cocycle $h: \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) \rightarrow \mathbb{Z}$ with finite image, and also the facts that both groups $\text{SL}_2(\mathbb{R})$ and $\widetilde{\text{SL}}_2(\mathbb{R})$ are perfect, i.e. equal to their commutator subgroups.

This led us to the following general question.

Question 1.1. When does an extension \widetilde{G} of a group G by an abelian group A satisfy $\widetilde{G}_B^{00} \neq \widetilde{G}_B^{000}$ for some parameter set B (working in a monster model)?

We would like to emphasize that we consider this problem in a general algebraic context, i.e. without assuming that \widetilde{G} is a topological universal cover of a topological group or that G is definable in an \mathcal{o} -minimal structure. The only restriction that we make is the assumption that the 2-cocycle $h: G \times G \rightarrow A$ defining our extension is definable and has finite image. Our goal is to find sufficient (and necessary, at least in some situations) conditions on h for which $\widetilde{G}_B^{00} \neq \widetilde{G}_B^{000}$.

In Section 2, we prove a theorem saying that $\widetilde{G}_B^{00} \neq \widetilde{G}_B^{000}$ under some general assumptions on the underlying 2-cocycle $h: G \times G \rightarrow A$, and we show that in some situations this assumption is also necessary. Using this theorem, we obtain the examples from [3] as well as certain new classes of examples of extensions for which $\widetilde{G}_B^{00} \neq \widetilde{G}_B^{000}$, e.g. some central extensions of $\text{SL}_2(k)$ for k being any ordered field. In order to apply our theorem to get these new examples, we use Matsumoto-Moore theory.

Knowing already that examples of groups G for which G^{00} and G^{000} differ exist, a natural question arises on how much they can differ. More formally, what can be said about G^{00}/G^{000} ? In all examples in [3] and also for all groups definable in \mathcal{o} -minimal expansions of real closed fields [4, Remark 5.5], this quotient turns out to be abelian. Therefore, the following problem was formulated in [3].

Problem 1.2. Find a (saturated) group G for which the quotient $G_\emptyset^{00}/G_\emptyset^{000}$ is not abelian.

In concrete examples that we have found in Section 4, this quotient is abelian, but we hope that our theorem from Section 2 may lead to a non-abelian example. A short discussion about it can be found around Question 2.6. In any case, in Section 2, we notice that whenever $G_B^{000} = G_B^{00}$, the quotient $\widetilde{G}_B^{00}/\widetilde{G}_B^{000}$ is abelian, and we give its description.

The relevant definitions and facts concerning extensions of groups and 2-cocycles are given in the initial parts of Sections 2 and 4. Here we only recall a few definitions and facts from model theory.

Suppose a set X is type-definable in a monster model. Let E be a bounded (i.e. with boundedly many classes), type-definable equivalence relation on X . The *logic topology* on X/E is a topology whose closed sets are the sets with type-definable preimages by the quotient map. It turns out that this is always a compact (Hausdorff) topology. In particular, the equivalence relation on G of lying in the same coset modulo G_B^{00} is always bounded and type-definable, so we have the logic topology on G/G_B^{00} . With this topology G/G_B^{00} becomes a compact, topological group. For the basic properties of the logic topology see for example [13, Section 2] and [7, Proposition 3.5 1.].

We often use, in various theorems and proofs of the present paper, the notion of an absolutely connected group G from [5]. That is, G is absolutely connected if $G^{*000} = G^*$, for an arbitrary monster model G^* of G . Here is a precise definition.

Definition 1.3 ([5, Definition 2.6, Proposition 2.5]). Let G be an infinite group.

- (1) Suppose G is given with some first order structure $\mathcal{G} = (G, \cdot, \dots)$. We say that \mathcal{G} is *definably absolutely connected* if \mathcal{G}^{*000} exists and $\mathcal{G}^* = \mathcal{G}^{*000}$ for any (equivalently some) monster model $\mathcal{G}^* \succ \mathcal{G}$.
- (2) We say that G is *absolutely connected* if for an arbitrary first order expansion $\mathcal{G} = (G, \cdot, \dots)$ of G , the component \mathcal{G}^{*000} exists and $\mathcal{G}^* = \mathcal{G}^{*000}$, where \mathcal{G}^* is a monster model.

Using certain limit process for the class of absolutely connected groups, another example of a group G for which $G_B^{00} \neq G_B^{000}$ has been given in [5, Proposition 2.13, Proposition 4.4] (in [5], the symbol G_B^∞ has been used to denote G_B^{000}). In fact, the aforementioned G is a countable direct product \mathbb{F}_ω^ω of the free group \mathbb{F}_ω , considered as a first order structure with a certain countable family of predicates.

The main examples of absolutely connected groups are Chevalley groups [5, Theorem. 3.14]. That is, if k is an arbitrary infinite field and G is a k -split, semisimple, simply connected linear algebraic group defined over k , then the group $G(k)$ of k -rational points is absolutely connected. In particular, all classical groups such as special linear groups $\mathrm{SL}_n(k)$ or symplectic groups $\mathrm{Sp}_n(k)$ are absolutely connected.

We will use the fact that absolutely connected groups are perfect [5, Theorem 2.16]. In some arguments, we also consider absolutely connected groups of finite commutator width (see the discussion before Proposition 2.9). Notice that all groups of the form $G(k)$ from the previous paragraph have finite commutator width (in fact, they are *weakly simple*, see [5, Proposition 2.19, Theorem 3.11]).

A few observations in this paper rely on a version of Beth's definability theorem for types, rather than for formulas. For completeness we give a proof, which is based on the proof of Beth's theorem for formulas [8, Theorem 2.2.22].

Let L be a first order language. Recall that for a first order L -theory T and collections of formulas $p_1(x)$ and $p_2(x)$ in the language L , the expression $T \models p_1(x) \equiv p_2(x)$ means that for any model $M \models T$ the types $p_1(x)$ and $p_2(x)$ have the same sets of realizations in M , i.e. $p_1(M) = p_2(M)$. Equivalently, for every $\varphi_1(x) \in p_1(x)$ there exists a conjunction $\varphi_2(x)$ of formulas from $p_2(x)$ such that $T \vdash \varphi_2(x) \rightarrow \varphi_1(x)$, and conversely, for every $\psi_2(x) \in p_2(x)$ there exists a conjunction $\psi_1(x)$ of formulas from $p_1(x)$ such that $T \vdash \psi_1(x) \rightarrow \psi_2(x)$.

Fact 1.4 (Beth's theorem for types). *Assume L is a first order language. Let $L' = L \cup \{P'_i, f'_j, c'_k : i \in I, j \in J, k \in K\}$ and $L'' = L \cup \{P''_i, f''_j, c''_k : i \in I, j \in J, k \in K\}$, where $P'_i, P''_i, f'_j, f''_j, c'_k, c''_k$ are pairwise distinct relation, function and constant symbols, not appearing in the language L , and such that for each $i \in I$ the relation symbols P'_i and P''_i have the same arity and for each $j \in J$ the function symbols f'_j and f''_j have the same arity. Let T' be a theory in L' and T'' the corresponding theory in L'' (i.e. T'' is obtained from T' by replacing each of the symbols P'_i, f'_j, c'_k by P''_i, f''_j, c''_k , respectively). Let $\pi'(x)$ be a collection of formulas in L' and $\pi''(x)$ the corresponding collection of formulas in L'' . Assume $T' \cup T'' \models \pi'(x) \equiv \pi''(x)$. Then there exists a collection of formulas $\pi(x)$ in the language L such that $T' \models \pi'(x) \equiv \pi(x)$.*

Proof. Denote the length of the tuple x by n . Let $d = (d_1, \dots, d_n)$ be a tuple of new constant symbols. Consider any $\alpha''(x) \in \pi''(x)$. By assumption, we can choose a conjunction $\beta'_{\alpha''}(x)$ of formulas from $\pi'(x)$ so that $T' \cup T'' \vdash \beta'_{\alpha''}(d) \rightarrow \alpha''(d)$. Thus, there exists a sentence $\varphi' \in T'$ such that for the corresponding sentence $\varphi'' \in T''$ we have $\varphi' \wedge \varphi'' \vdash \beta'_{\alpha''}(d) \rightarrow \alpha''(d)$ (we may assume that T' is closed under conjunctions). Therefore,

$$\varphi' \wedge \beta'_{\alpha''}(d) \vdash \varphi'' \rightarrow \alpha''(d).$$

Since the left hand side is a sentence in $L' \cup \{d_1, \dots, d_n\}$ and the right hand side is a sentence in $L'' \cup \{d_1, \dots, d_n\}$, by Craig Interpolation Theorem (see [8, Theorem 2.2.20]), there exists a formula $\psi_{\alpha''}(x)$ in the language L such that

$$\varphi' \wedge \beta'_{\alpha''}(d) \vdash \psi_{\alpha''}(d) \text{ and } \psi_{\alpha''}(d) \vdash \varphi'' \rightarrow \alpha''(d).$$

As any model for $L' \cup \{d_1, \dots, d_n\}$ is clearly a model for $L'' \cup \{d_1, \dots, d_n\}$ with the symbols P''_i, f''_j, c''_k interpreted in the same way as the symbols P'_i, f'_j, c'_k , we get

$$\psi_{\alpha''}(d) \vdash \varphi' \rightarrow \alpha'(d),$$

where $\alpha'(x) \in T'$ corresponds to $\alpha''(x)$.

We conclude that

$$\varphi' \vdash (\beta'_{\alpha''}(d) \rightarrow \psi_{\alpha''}(d)) \wedge (\psi_{\alpha''}(d) \rightarrow \alpha'(d)),$$

and so

$$\varphi' \vdash (\forall x)(\beta'_{\alpha''}(x) \rightarrow \psi_{\alpha''}(x)) \wedge (\forall x)(\psi_{\alpha''}(x) \rightarrow \alpha'(x)).$$

From this, we see that

$$T' \models \pi'(x) \equiv \{\psi_{\alpha''}(x) : \alpha''(x) \in \pi''(x)\},$$

and $\{\psi_{\alpha''}(x) : \alpha''(x) \in \pi''(x)\}$ is a collection of formulas in L . □

2. CONNECTED COMPONENTS AND 2-COCYCLES

This section is constructed as follows. After a short introduction on group extensions and 2-cocycles, we prove the main theorem of this paper (Theorem 2.1), describing a general situation in which an extension \widetilde{G} of a group G by an abelian group A , defined in terms of a definable 2-cocycle with finite image, satisfies $\widetilde{G}^{*000}_B \neq \widetilde{G}^{*00}_B$ for some parameter set B (* means that we consider the sets of realizations in a monster model). Then, we notice that the examples from [3] fit into this context. Next, we make a closer analysis of the situation from Theorem 2.1, proving that in some cases the main assumption of this theorem is also a necessary condition for $\widetilde{G}^{*000}_B \neq \widetilde{G}^{*00}_B$, and

describing the quotient $\widetilde{G}_B^{*00}/\widetilde{G}_B^{*00}$. In Section 3, we prove that for some situations in which Theorem 2.1 (or its Corollary 2.2) can be applied, it can also be applied to certain extensions of the underlying group G . In Section 4, the above results are used to produce new classes of examples of groups of the form \widetilde{G}^* satisfying $\widetilde{G}_B^{*00} \neq \widetilde{G}_B^{*00}$.

Let G be an arbitrary group and let A be an abelian group. Assume that \widetilde{G} is an extension of G by A (not necessarily central), i.e. we assume that there exists an exact sequence

$$(1) \quad 1 \hookrightarrow A \hookrightarrow \widetilde{G} \xrightarrow{\pi} G \twoheadrightarrow 1.$$

Sometimes by a group extension of G by A we mean the above sequence (and not just the group \widetilde{G}), which should be clear from the context.

Then, $\widetilde{G} \cong (A \times G, *)$, where

$$(a_1, g_1) * (a_2, g_2) = (a_1 + g_1 \cdot a_2 + h(g_1, g_2), g_1 g_2)$$

for some action $\cdot : G \times A \rightarrow A$ of G on A by automorphisms such that the conjugation action of \widetilde{G} on A induces \cdot , and for some 2-cocycle h (also called a *factor set*), that is a map $h : G \times G \rightarrow A$ satisfying [16, 10.13]:

- $h(g_1, g_2) + h(g_1 g_2, g_3) = h(g_1, g_2 g_3) + g_1 \cdot h(g_2, g_3)$ for all $g_1, g_2, g_3 \in G$,
- $h(g, e) = h(e, g) = 0$ for all $g \in G$.

The identity element is $(0, e)$ and the inverse of (a, g) equals $(-g^{-1} \cdot a - h(g^{-1}, g), g^{-1})$.

More precisely, the extension (1) is equivalent to the natural extension

$$(2) \quad 1 \hookrightarrow A \hookrightarrow (A \times G, *) \xrightarrow{\pi} G \twoheadrightarrow 1,$$

that is, there exists an isomorphism $\varphi : \widetilde{G} \xrightarrow{\cong} (A \times G, *)$ such that the following diagram commutes

$$\begin{array}{ccccc} A & \hookrightarrow & \widetilde{G} & \xrightarrow{\pi} & G \\ \downarrow \text{id} & & \downarrow \varphi \cong & & \downarrow \text{id} \\ A & \hookrightarrow & (A \times G, *) & \xrightarrow{\pi'} & G. \end{array}$$

Conversely, for any 2-cocycle $h : G \times G \rightarrow A$, the structure $(A \times G, *)$ is a group being an extension of G by A .

From now on, \widetilde{G} will denote $(A \times G, *)$ for some 2-cocycle h . We will freely identify A with the subgroup $A \times \{e\}$ of \widetilde{G} .

We say that the 2-cocycle $h : G \times G \rightarrow A$ is *splitting* if there exists a function $f : G \rightarrow A$ for which:

- $h(g_1, g_2) = f(g_1) + g_1 \cdot f(g_2) - f(g_1 g_2)$ for all $g_1, g_2 \in G$,
- $f(e) = 0$.

In this situation, we also say that h is *splitting via the function f* .

We say that 2-cocycles h and h' are *cohomologous* via a function f if $h - h'$ is splitting via f .

Recall that the 2-cocycle h is splitting if and only if the extension (2) is equivalent to the semidirect product extension of G by A [16, 10.15]. In particular, if the action of G on A is trivial, then h is splitting if and only if the extension (2) is equivalent to the product extension.

If $H \leq G$, then h induces the restricted 2-cocycle $h|_{H \times H}: H \times H \rightarrow A$. If $C \leq A$ is invariant under the action of G , then the action of G on A induces an action of G on A/C , and the 2-cocycle h induces a 2-cocycle $\bar{h}: G \times G \rightarrow A/C$ in an obvious way.

We consider a situation when the groups G and A and the action of G on A are \emptyset -definable in a (many-sorted) structure \mathcal{G} (e.g. \mathcal{G} consists of the pure groups G and $(A, +)$ together with the action of G on A). We assume that the image $\text{Im}(h)$ of the 2-cocycle $h: G \times G \rightarrow A$ is finite and that h is definable in \mathcal{G} (equivalently, the preimage by h of any element of A is a definable in \mathcal{G} subset of $G \times G$).

The group \tilde{G} is, of course, definable in \mathcal{G} . Let $\mathcal{G}^* \succ \mathcal{G}$ be a monster model. Denote by G^* the interpretation of G in \mathcal{G}^* , by A^* the interpretation of A in \mathcal{G}^* , and by \tilde{G}^* the interpretation of \tilde{G} in \mathcal{G}^* . We have the following exact sequence

$$(3) \quad 1 \hookrightarrow A^* \hookrightarrow \tilde{G}^* \xrightarrow{\pi} G^* \twoheadrightarrow 1,$$

where π is the projection on the second coordinate.

The interpretations in \mathcal{G}^* of various definable objects will be usually denoted by adding $*$ (as above). An exception is the 2-cocycle h whose interpretation in \mathcal{G}^* will be denoted by h (and not by h^*).

In the next theorem, by G_B^{*000} and \tilde{G}_B^{*000} we denote the smallest subgroups of bounded index of G^* and \tilde{G}^* , respectively, which are invariant under $\text{Aut}(\mathcal{G}^*/B)$ for a fixed parameter set B . By G_B^{*00} and \tilde{G}_B^{*00} we denote the smallest subgroups of bounded index of G^* and \tilde{G}^* , respectively, which are type-definable in \mathcal{G}^* over B . By A_0 we will denote the subgroup of A generated by the image of h . Notice that A_0 is countable.

Theorem 2.1. *Let G be a group acting by automorphisms on an abelian group A , where G , A and the action of G on A are \emptyset -definable in a structure \mathcal{G} , and let $h: G \times G \rightarrow A$ be a 2-cocycle which is B -definable in \mathcal{G} and with finite image $\text{Im}(h)$ contained in B for some finite parameter set $B \subset \mathcal{G}$. Let A_1^* be a bounded index subgroup of A^* which is type-definable over B and which is invariant under the action of G^* . Assume that:*

- (i) *the 2-cocycle $h|_{G_B^{*00} \times G_B^{*00}}: G_B^{*00} \times G_B^{*00} \rightarrow A_0$ is not cohomologous to a 2-cocycle which takes values in $A_1^* \cap A_0$ (via a function taking values in A_0); equivalently, the induced 2-cocycle $\bar{h}: G_B^{*00} \times G_B^{*00} \rightarrow A_0 / (A_1^* \cap A_0)$ is non-splitting,*
- (ii) *$A_0 / (A_1^* \cap A_0)$ is torsion free (and so isomorphic with \mathbb{Z}^n for some natural n).*

*Then $\tilde{G}_B^{*000} \neq \tilde{G}_B^{*00}$.*

*Suppose furthermore that $G_B^{*000} = G^*$, and for every proper, type-definable over B in \mathcal{G}^* and invariant under the action of G^* subgroup H of A^* with bounded index the induced 2-cocycle $\bar{h}: G^* \times G^* \rightarrow A_0 / (H \cap A_0)$ is non-splitting. Then $\tilde{G}_B^{*00} = \tilde{G}^*$.*

Before the proof, let us first list two special cases in which the assumption that A_1^* is invariant under G^* is satisfied.

- Consider the case when $A_1^* = A^{*0}$ is the connected component of the pure group $(A^*, +)$, i.e. it is the intersection of all definable (in the pure group $(A^*, +)$) finite index subgroups of A^* . Then A_1^* has bounded index in A^* , it is type-definable in \mathcal{G}^* over \emptyset , and it is invariant under the action of G^* .
- If the action of G on A is trivial (i.e. the extension of G by A is central), then the assumption that A_1^* is invariant under G^* is clearly satisfied for every A_1^* .

Notice also that if the assumptions of the theorem are satisfied in the monster model \mathcal{G}^* , then they are satisfied in any bigger monster model.

Proof. First, we prove the following claim.

Claim 1. Suppose $H \leq A^*$ is a subgroup of bounded index which is invariant both under $\text{Aut}(\mathcal{G}^*/B)$ and under the action of G^* on A^* . Then $(H + A_0) \times G_B^{*000}$ is a subgroup of \widetilde{G}^* containing \widetilde{G}_B^{*000} .

Proof of Claim 1. The fact that $(H + A_0) \times G_B^{*000}$ is a subgroup of \widetilde{G}^* follows from the invariance of H under the action of G^* and the observations that the image $\text{Im}(h)$ is contained in A_0 and A_0 is closed under the action of G^* (which follows from the first formula in the definition of 2-cocycles). Moreover, it is clear that $(H + A_0) \times G_B^{*000}$ is B -invariant and has bounded index in \widetilde{G}^* . This shows the desired inclusion. \square

It is easy to see that $G_B^{*000} = \pi \left[\widetilde{G}_B^{*000} \right]$ (in (3)). Let $H \leq A^*$ be as in Claim 1. As a consequence of Claim 1, we have that the following sequences are exact:

$$(4) \quad 1 \hookrightarrow H + A_0 \hookrightarrow (H + A_0) \times G_B^{*000} \xrightarrow{\pi} G_B^{*000} \twoheadrightarrow 1,$$

$$1 \hookrightarrow (H + A_0) \cap \widetilde{G}_B^{*000} \hookrightarrow \widetilde{G}_B^{*000} \xrightarrow{\pi} G_B^{*000} \twoheadrightarrow 1.$$

We can say even more about this situation. Notice that if H satisfies the assumptions of Claim 1, then so does $H \cap \widetilde{G}_B^{*000}$, because \widetilde{G}_B^{*000} is a normal subgroup of \widetilde{G}^* . Thus, Claim 1 yields the following exact sequence

$$(5) \quad 1 \hookrightarrow \left(H \cap \widetilde{G}_B^{*000} + A_0 \right) \cap \widetilde{G}_B^{*000} \hookrightarrow \widetilde{G}_B^{*000} \xrightarrow{\pi} G_B^{*000} \twoheadrightarrow 1,$$

and so $\ker \left(\pi|_{\widetilde{G}_B^{*000}} \right) = H \cap \widetilde{G}_B^{*000} + A_0 \cap \widetilde{G}_B^{*000}$.

Claim 2. Let $H \leq A^*$ be as in Claim 1. If $A_0 \cap \widetilde{G}_B^{*000} \subseteq H$, then the induced 2-cocycle $\bar{h}: G_B^{*000} \times G_B^{*000} \rightarrow A_0/(H \cap A_0)$ is splitting.

Proof of Claim 2. If $A_0 \cap \widetilde{G}_B^{*000} \subseteq H$, then the above conclusion gives us

$$\ker \left(\pi|_{\widetilde{G}_B^{*000}} \right) = H \cap \widetilde{G}_B^{*000}.$$

Let $s: G_B^{*000} \rightarrow \widetilde{G}_B^{*000}$ be a section of π of the form $s(g) = (a_g, g)$, where each $a_g \in H + A_0$ and $a_e = 0$. Then,

$$a_g = b_g + c_g,$$

for some $b_g \in H$ and $c_g \in A_0$. Consider a 2-cocycle $h': G_B^{*000} \times G_B^{*000} \rightarrow A^*$ defined by

$$h'(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1}.$$

It takes values in $\ker \left(\pi|_{\widetilde{G}_B^{*000}} \right) = H \cap \widetilde{G}_B^{*000}$. Moreover, it is cohomologous to h , because, using the fact that $g \cdot h(g^{-1}, g) = h(g, g^{-1})$ (which follows from the definition of 2-cocycles), we have

$$\begin{aligned} h'(g_1, g_2) &= (a_{g_1}, g_1)(a_{g_2}, g_2)(a_{g_1g_2}, g_1g_2)^{-1} \\ &= h(g_1, g_2) + g_1 \cdot a_{g_2} - a_{g_1g_2} + a_{g_1} \\ &\quad + h(g_1g_2, (g_1g_2)^{-1}) - g_1g_2 \cdot h((g_1g_2)^{-1}, g_1g_2) \\ &= h(g_1, g_2) + g_1 \cdot a_{g_2} - a_{g_1g_2} + a_{g_1}. \end{aligned}$$

The above equality implies that

$$h'(g_1, g_2) - g_1 \cdot b_{g_2} + b_{g_1g_2} - b_{g_1} = h(g_1, g_2) + g_1 \cdot c_{g_2} - c_{g_1g_2} + c_{g_1}.$$

Both sides of this equality define the same 2-cocycle $h''(g_1, g_2)$, which takes values in $H \cap A_0$ (because the left hand side takes values in H , and the right hand side takes values in A_0). Thus, $h|_{G_B^{*000} \times G_B^{*000}}$ is cohomologous to a cocycle with values in $H \cap A_0$ via the function $f: G_B^{*000} \rightarrow A_0$ defined by $f(g) = -c_g$. \square

To prove the first part of the theorem, suppose for a contradiction that $\widetilde{G}_B^{*000} = \widetilde{G}_B^{*00}$. Put $H = \widetilde{G}_B^{*000} \cap A^*_1$. Then, H is a subgroup of bounded index of A^* , which is type-definable in \mathcal{G}^* over B and invariant under the action of G^* (the last fact follows from the observation that \widetilde{G}_B^{*000} is a normal subgroup of \widetilde{G}^*). By Claim 1, we get that $(H + A_0) \times G_B^{*000}$ is a group containing \widetilde{G}_B^{*000} . Thus, $\widetilde{G}_B^{*000} \cap A^* \leq H + A_0$. On the other hand, by Claim 2 together with the assumption (i) and the fact that $G_B^{*00} = \pi \left[\widetilde{G}_B^{*00} \right] = \pi \left[\widetilde{G}_B^{*000} \right] = G_B^{*000}$, we get $A_0 \cap \widetilde{G}_B^{*000} \not\subseteq H$, so $\widetilde{G}_B^{*000} \cap ((H + A_0) \setminus H) \neq \emptyset$. Using the assumption that $A_0 / (A^*_1 \cap A_0)$ is torsion free and the fact that A_0 is countable, we conclude that $(\widetilde{G}_B^{*000} \cap A^*) / H$ is countably infinite. But, since $\widetilde{G}_B^{*000} = \widetilde{G}_B^{*00}$, the group $\widetilde{G}_B^{*000} \cap A^*$ is type-definable in \mathcal{G}^* , and so $(\widetilde{G}_B^{*000} \cap A^*) / H$ is a compact topological group, and as such, it cannot be of cardinality \aleph_0 , a contradiction.

Now, we prove the second part of the theorem. Let $H = \widetilde{G}_B^{*00} \cap A^*$. By Claim 2 and our assumption, $H = A^*$. Therefore, $A^* \leq \widetilde{G}_B^{*00}$. On the other hand $\pi \left[\widetilde{G}_B^{*00} \right] \supseteq \pi \left[\widetilde{G}_B^{*000} \right] = G_B^{*000} = G^*$. Hence, $\widetilde{G}_B^{*00} = \widetilde{G}^*$. \square

Corollary 2.2. *Let G be a group acting by automorphisms on an abelian group A , where G , A and the action of G on A are \emptyset -definable in a structure \mathcal{G} , and let $h: G \times G \rightarrow A$ be a 2-cocycle which is B -definable in \mathcal{G} and with finite image $\text{Im}(h)$ contained in B for some finite parameter set $B \subset \mathcal{G}$. Recall that A_0 denotes the subgroup of A generated by $\text{Im}(h)$. Let A^*_1 be a bounded index subgroup of A^* which is type-definable (in \mathcal{G}^*) over B and which is invariant under the action of G^* . Assume that:*

- (1) *The 2-cocycle $h: G \times G \rightarrow A_0$ is non-splitting (via a function taking values in A_0).*
- (2) *$A^*_1 \cap A_0$ is trivial and A_0 is torsion free (and so $A_0 \cong \mathbb{Z}^n$ for some n).*
- (3) *$G_B^{*00} = G^*$.*

Then $\widetilde{G}_B^{*000} \neq \widetilde{G}_B^{*00}$.

Suppose furthermore that $G_B^{*000} = G^*$ and:

- (4) (strong non-splittingness of h) For every proper subgroup $Z \subsetneq A_0 = \mathbb{Z}^n$ the induced 2-cocycle $\bar{h}: G \times G \rightarrow \mathbb{Z}^n/Z$ is non-splitting.
- (5) (denseness of A_0 in A^*) If $H \leq A^*$ is a type-definable over B (in \mathcal{G}^*) subgroup of A^* of bounded index and containing A_0 , then $H = A^*$.

Then $\widetilde{G}_B^{*00} = \widetilde{G}^*$.

Before the proof, we give some comments on the assumptions of this corollary. Notice that since A_0 is finitely generated, (2) implies that $A_0 \cong \mathbb{Z}^n$ for some natural number n . By Remark 2.7(i), every subgroup Z considered in (4) is invariant under the action of G^* , and so it makes sense to talk about the induced 2-cocycle $\bar{h}: G \times G \rightarrow \mathbb{Z}^n/Z$. In this paper, strong non-splittingness (in the sense of (4)) of some 2-cocycles will be achieved by using Corollary 4.8. Another remark is that every definably absolutely connected group (see Definition 1.3) satisfies (3). Finally, we explain why (5) was called ‘denseness of A_0 in A^* ’. Let A_B^{*00} be the smallest subgroup of bounded index of A^* which is type-definable over B in \mathcal{G}^* . It is easy to check that (5) is equivalent to the fact that A_0/A_B^{*00} is a dense subset of the topological group A^*/A_B^{*00} .

Proof. We have to prove that the assumptions of Theorem 2.1 are satisfied. First note that by (2), the group $A_0/(A_1^* \cap A_0) = A_0$ is torsion free. Suppose for a contradiction that the 2-cocycle $h: G^* \times G^* \rightarrow A_0$ is splitting (we use here (2) and (3)). Then, after restriction to G , we get that $h: G \times G \rightarrow A_0$ is splitting (via a function taking values in A_0), a contradiction with (1).

The second part of the corollary holds, because (4) and (5) imply that for every proper, type-definable over B (in \mathcal{G}^*) subgroup H of A^* of bounded index the induced 2-cocycle $\bar{h}: G \times G \rightarrow A_0/(H \cap A_0)$ is non-splitting. \square

Next, we notice that the examples from [3, Section 3] follow from the above corollary.

Example 2.3. Let $\mathcal{G} = ((\mathbb{Z}, +), (\mathbb{R}, +, \cdot, <, 0, 1))$, $G = \mathrm{SL}_2(\mathbb{R})$ and $A = (\mathbb{Z}, +)$. The groups G and A are \emptyset -definable in \mathcal{G} . Assume that the action of G on A is trivial. Let $\widetilde{G} = \mathrm{SL}_2(\mathbb{R})$ be the topological universal cover of $\mathrm{SL}_2(\mathbb{R})$. $\mathrm{SL}_2(\mathbb{R})$ is defined by means of the 2-cocycle $h: G \times G \rightarrow \mathbb{Z}$ considered in [1, Theorem 2]. For the reader’s convenience we recall the definition of h from [1], from which it is clear that h is B -definable in \mathcal{G} , where

$B := \{-1, 0, 1\}$. For $c, d \in \mathbb{R}$ define the following symbol $c(d) = \begin{cases} c & : c \neq 0 \\ d & : c = 0 \end{cases}$. Consider any $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. Let $\begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$. Then

$$h\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) = \begin{cases} 1 & : c_1(d_1) > 0 \wedge c_2(d_2) > 0 \wedge c_3(d_3) < 0 \\ -1 & : c_1(d_1) < 0 \wedge c_2(d_2) < 0 \wedge c_3(d_3) > 0 \\ 0 & : \text{otherwise} \end{cases}.$$

Put as A_1^* the connected component \mathbb{Z}^{*0} of the pure group $(\mathbb{Z}^*, +)$. Then the assumptions of the above corollary are satisfied, so we get $\widetilde{G}^* = \widetilde{G}^{*00} \neq \widetilde{G}^{*000}$, where \widetilde{G}^* is the interpretation of \widetilde{G} in a monster model $\mathcal{G}^* = ((\mathbb{Z}^*, +), (\mathbb{R}^*, +, \cdot, <, 0, 1))$.

Proof. The assumptions (1) and (4) of Corollary 2.2 are true by [1, Theorem 1], (3) follows from the absolute connectedness of $\mathrm{SL}_2(\mathbb{R})$, and $A_0 = \mathbb{Z}$ implies (2). The condition (5) follows from the observation that if $D \subseteq \mathbb{Z}^*$ is a B -definable subset containing \mathbb{Z} and $B \subseteq \mathbb{Z}$, then $D = \mathbb{Z}^*$. Notice that since \mathcal{G} has NIP, we can skip the parameter set B in all connected components. \square

Example 2.4. Let $\mathcal{G} = (\mathbb{R}, +, \cdot, <, 0, 1)$, $G = \mathrm{SL}_2(\mathbb{R})$ and $A = \mathrm{SO}_2(\mathbb{R})$. Assume that the action of G on A is trivial. Fix a non-torsion element $g \in A$. Let \tilde{G} be defined by means of the 2-cocycle $h': G \times G \rightarrow \mathrm{SO}_2(\mathbb{R})$ defined as $h'(x, y) = h(x, y)g$ for h from Example 2.3. Put $A_1^* = \mathrm{SO}_2(\mathbb{R}^*)^{00}$ and $B = \{-g, 0, g\}$. Then, as in the previous example, the assumptions of Corollary 2.2 are satisfied, so we get $\tilde{G}^* = \tilde{G}^{*00} \neq \tilde{G}^{*000}$.

The advantage of our argument (i.e. the proof of Theorem 2.1) is that it is more general than the argument in [3]. In particular, it does not use the facts that G and \tilde{G} are perfect. In Section 4, we will see that Theorem 2.1 yields new classes of examples of groups \tilde{G} for which $\tilde{G}^{*00}_B \neq \tilde{G}^{*000}_B$, and we hope it may lead to more such examples.

In Examples 2.3 and 2.4, the quotient group $\tilde{G}^{*00}/\tilde{G}^{*000}$ turns out to be abelian. The next, general remark shows that this is not accidental.

Remark 2.5. In the situation from the first part of Theorem 2.1, if $G^{*00}_B = G^{*000}_B$, then the quotient $\tilde{G}^{*00}_B/\tilde{G}^{*000}_B$ is abelian.

Proof. We easily check that $\pi \left[\tilde{G}^{*00}_B \right] = G^{*000}_B$ and $\pi \left[\tilde{G}^{*00}_B \right] = G^{*00}_B$, where $\pi: \tilde{G}^* \rightarrow G^*$ is the projection on the second coordinate. Take any $(a, g) \in \tilde{G}^{*00}_B$. By assumption, there is $a' \in A^*$ such that $(a', g) \in \tilde{G}^{*000}_B$. Since $(a, g)(a', g)^{-1} = (a - a', e) \in A^*$, we get that $(a, g)\tilde{G}^{*000}_B \in A^*/\tilde{G}^{*000}_B$. Hence, $\tilde{G}^{*00}_B/\tilde{G}^{*000}_B \subseteq A^*/\tilde{G}^{*000}_B$, so the desired conclusion follows from the assumption that A^* is abelian. \square

A question arises whether one can solve Problem 1.2 by applying Theorem 2.1. In virtue of Remark 2.5, we see that in order to do that, we cannot apply Theorem 2.1 to the situation when $G^{*00}_B = G^{*000}_B$. However, one could try to apply this theorem to groups for which $G^{*00}_B \neq G^{*000}_B$. One should notice here that if $G^{*00}_B \neq G^{*000}_B$, then $\tilde{G}^{*000}_B \neq \tilde{G}^{*00}_B$ is clear without using Theorem 2.1. Nevertheless, the proof of Theorem 2.1 explains a deeper reason why $\tilde{G}^{*000}_B \neq \tilde{G}^{*00}_B$, and it may help to construct an example in which $\tilde{G}^{*00}_B/\tilde{G}^{*000}_B$ is non-abelian. Let us state our question explicitly.

Question 2.6. Can we find data satisfying the assumptions of Theorem 2.1 (necessarily with $G^{*00}_B \neq G^{*000}_B$) so that $\tilde{G}^{*00}_B/\tilde{G}^{*000}_B$ is not abelian? Does the answer change under the additional assumption that the action of G on A is trivial?

Maybe one could try to iterate the applications of Theorem 2.1 in an appropriate way to get the desired example at some point of the construction.

Now, we will undertake a closer analysis of the situation from Theorem 2.1. First of all, we show that in some cases the assumption (i) of Theorem 2.1 is not only sufficient but also necessary in order to have $\tilde{G}^{*000}_B \neq \tilde{G}^{*00}_B$. Then, we give a description of the quotient $\tilde{G}^{*00}_B/\tilde{G}^{*000}_B$, assuming that $G^{*000}_B = G^{*00}_B$. We also formulate some questions related to these issues.

Before we go to the details, notice that if one wants to find a necessary condition on the 2-cocycle h from Theorem 2.1 for \widetilde{G}^* to satisfy $\widetilde{G}_{B}^{*000} \neq \widetilde{G}_{B}^{*00}$, it is natural to assume that $G_B^{*000} = G_B^{*00}$ (as otherwise \widetilde{G}_{B}^{*000} is automatically different from \widetilde{G}_{B}^{*00}).

Let us start from the following, very useful observation.

Remark 2.7. Consider the situation described in the first sentence of Theorem 2.1.

- (i) G_B^{*0} acts trivially on A_0 .
- (ii) G_B^{*000} acts trivially on $A^* / (\widetilde{G}_{B}^{*000} \cap A^*)$.
- (iii) Let A_1^* be a bounded index subgroup of A^* which is type-definable over B and which is invariant under the action of G^* . Then G_B^{*00} acts trivially on A^* / A_1^* .

Proof. (i) Take any $a \in A_0$. Since A_0 is closed under the action of G^* and countable, the index $[G^* : \text{Stab}_{G^*}(a)]$ is also countable, where $\text{Stab}_{G^*}(a)$ is the stabilizer of a in G^* . As $a \in \text{dcl}(B)$, we have that $\text{Stab}_{G^*}(a)$ is B -definable with $[G^* : \text{Stab}_{G^*}(a)]$ finite, and so $\text{Stab}_{G^*}(a) \geq G_B^{*0}$. Thus, G_B^{*0} acts trivially on A_0 .

(ii) Consider any element \bar{a} in $A^* / (\widetilde{G}_{B}^{*000} \cap A^*)$. Since \widetilde{G}_{B}^{*000} is a normal subgroup of \widetilde{G}^* , the action of G^* on A^* induces a B -invariant action of G^* on the quotient $A^* / (\widetilde{G}_{B}^{*000} \cap A^*)$. As this quotient has bounded size, the orbit of \bar{a} under the action of G^* is also of bounded size. Thus, the stabilizer $\text{Stab}_{G^*}(\bar{a})$ of \bar{a} in G^* is a bounded index subgroup of G^* (invariant over B, \bar{a}). The intersection of all such stabilizers $\text{Stab}_{G^*}(\bar{a})$ for \bar{a} ranging over $A^* / (\widetilde{G}_{B}^{*000} \cap A^*)$ is a B -invariant, bounded index subgroup of G^* , and so it contains G_B^{*000} . Thus, G_B^{*000} acts trivially on $A^* / (\widetilde{G}_{B}^{*000} \cap A^*)$.

(iii) A similar proof works. □

Proposition 2.8. Consider the situation described in the first two sentences of Theorem 2.1 together with the assumption (ii) of this theorem, and suppose $G_B^{*000} = G_B^{*00}$. Assume additionally that $A_1^* \subseteq \widetilde{G}_{B}^{*000} \cap A^*$. Then $\widetilde{G}_{B}^{*000} \neq \widetilde{G}_{B}^{*00}$ if and only if $\widetilde{G}_{B}^{*00} \cap A^* \not\subseteq A_1^*$.

More precisely, the implication (\Leftarrow) does not require the above extra assumptions (except those from Theorem 2.1), whereas the implication (\Rightarrow) does not require the assumption (ii) from Theorem 2.1.

Before the proof, notice that the assumption $A_1^* \subseteq \widetilde{G}_{B}^{*000} \cap A^*$ cannot be dropped. Indeed, consider $A_1^* = A^*$ to see that without this assumption the implication (\Rightarrow) does not need to hold. One can also take any example in which $\widetilde{G}_{B}^{*000} \neq \widetilde{G}_{B}^{*00}$ and define a new A_1^* as $\widetilde{G}_{B}^{*00} \cap A^*$.

Notice, however, that the assumption $A_1^* \subseteq \widetilde{G}_{B}^{*000} \cap A^*$ holds in various interesting cases (e.g. if $A_1^* = A_B^{*00} = A_B^{*000}$ and the action of G on A is trivial; in particular, if $\mathcal{G} = ((A, +), M)$, G is definable in the structure M and it acts trivially on A , and $A_1^* = A^{*0}$).

Proof. (\Leftarrow) follows by an argument similar to the one contained in the last but one paragraph of the proof of Theorem 2.1 (and does not require the extra assumptions made in Proposition 2.8). Namely, suppose for a contradiction that $\widetilde{G}_{B}^{*000} = \widetilde{G}_{B}^{*00}$. By Claim 1 in the proof of Theorem 2.1, we know that $\widetilde{G}_{B}^{*000} \cap A^* \subseteq A_1^* + A_0$, so $(\widetilde{G}_{B}^{*000} \cap A^*) / A_1^*$

is countable, and, if it is non-trivial, it must be infinite (as $(A_0 + A_1^*)/A_1^*$ is torsion free by (ii)). But $(\widetilde{G}_B^{*000} \cap A^*)/A_1^*$ is non-trivial, because $\widetilde{G}_B^{*000} \cap A^* = \widetilde{G}_B^{*00} \cap A^* \not\subseteq A_1^*$. Hence, $(\widetilde{G}_B^{*00} \cap A^*)/A_1^*$ is a countable, infinite, compact group, which is impossible.

(\Rightarrow). Suppose for a contradiction that $\widetilde{G}_B^{*00} \cap A^* \subseteq A_1^*$. Then $A_1^* = \widetilde{G}_B^{*00} \cap A^* = \widetilde{G}_B^{*000} \cap A^*$. Since $G_B^{*000} = G_B^{*00}$, by the exactness of the sequences

$$(6) \quad 1 \hookrightarrow \widetilde{G}_B^{*00} \cap A^* \hookrightarrow \widetilde{G}_B^{*00} \xrightarrow{\pi} G_B^{*00} \longrightarrow 1,$$

$$(7) \quad 1 \hookrightarrow \widetilde{G}_B^{*000} \cap A^* \hookrightarrow \widetilde{G}_B^{*000} \xrightarrow{\pi} G_B^{*000} \longrightarrow 1,$$

we get $\widetilde{G}_B^{*000} = \widetilde{G}_B^{*00}$, which is a contradiction. \square

The *commutator length* of an element $g \in [G, G]$ is the minimal number of commutators sufficient to express g as their product. The *commutator width* $\text{cw}(G)$ of G is the maximum of the commutator lengths of elements of its derived subgroup $[G, G]$. Notice that, by a compactness argument, the clause ‘ G^* is perfect’ is equivalent to ‘ G is perfect and G has finite commutator width $\text{cw}(G)$ ’. By [5, Theorem 2.16], every absolutely connected group is perfect.

Proposition 2.9. *Consider the situation described in the first two sentences of Theorem 2.1.*

(i) *Assume additionally that $\text{Hom}(G_B^{*000}, A_0/(A_1^* \cap A_0))$ is trivial. Then the induced 2-cocycle $\bar{h}: G_B^{*000} \times G_B^{*000} \rightarrow A_0/(A_1^* \cap A_0)$ is non-splitting if and only if $\widetilde{G}_B^{*000} \cap A^* \not\subseteq A_1^*$.*
(ii) *Assume additionally that G is absolutely connected of finite commutator width (and so $G_B^{*000} = G_B^{*00} = G^*$ is perfect). Then the induced 2-cocycle $\bar{h}: G_B^{*00} \times G_B^{*00} \rightarrow A_0/(A_1^* \cap A_0)$ is non-splitting if and only if $\widetilde{G}_B^{*00} \cap A^* \not\subseteq A_1^*$. These two equivalent conditions are also equivalent to the fact that the induced 2-cocycle from $G^* \times G^*$ to A^*/A_1^* (which we also denote by \bar{h}) is non-splitting.*

The assumption that $\text{Hom}(G_B^{*000}, A_0/(A_1^* \cap A_0))$ is trivial is satisfied in many cases. For instance, this assumption holds when G_B^{*000} is a perfect group or a divisible group, because $A_0/(A_1^* \cap A_0)$ is a finitely generated abelian group. For example, we know that $G_B^{*000} = G^*$ is perfect when G is absolutely connected of finite commutator width, and G_B^{*000} is divisible when G is a finitely generated abelian group [2, Proposition 3.7].

Proof. (i) (\Rightarrow) follows from Claim 2 in the proof of Theorem 2.1 (and does not require the assumption that $\text{Hom}(G_B^{*000}, A_0/(A_1^* \cap A_0))$ is trivial).

(\Leftarrow). Since $A_0/(A_1^* \cap A_0) \cong (A_0 + A_1^*)/A_1^*$, \bar{h} can be treated as a 2-cocycle with values in $(A_0 + A_1^*)/A_1^*$. Suppose for a contradiction that \bar{h} is splitting. By Remark 2.7(i), this means that $\bar{h}(x, y) = f(x) + f(y) - f(xy)$ for some function $f: G_B^{*000} \rightarrow (A_0 + A_1^*)/A_1^*$ satisfying $f(e) = 0 + A_1^*$.

Notice that f is unique. Indeed, if $\bar{h}(x, y) = f_1(x) + f_1(y) - f_1(xy)$, then $(f - f_1)(xy) = (f - f_1)(x) + (f - f_1)(y)$, so $f - f_1$ belongs to $\text{Hom}(G_B^{*000}, (A_0 + A_1^*)/A_1^*)$ which is trivial (this is the only place where this assumption is used). Thus, we get that $f = f_1$.

From the uniqueness of f , we conclude that f , treated as a subset of $G_B^{*000} \times (A_0 + A_1^*)/A_1^*$, is invariant over B .

Let H be the extension of G_B^{*000} by A^*/A_1^* defined by means of the 2-cocycle \bar{h} , with the action of G_B^{*000} on A^*/A_1^* induced by the action of G^* on A^* , which is trivial by Remark 2.7(iii). Let K be the corresponding product $A^*/A_1^* \times G_B^{*000}$. Both groups H and K live in \mathcal{G}^* as B -invariant objects.

A classical fact tells us that since \bar{h} is splitting via f , the function $\Phi: H \rightarrow K$ given by $\Phi(a, g) = (a + f(g), g)$ is an isomorphism of groups. Since f is B -invariant, so is Φ .

Although K is not definable but only B -invariant, we can still define K_B^{000} as the smallest B -invariant, bounded index subgroup of K . The trivial subgroup A_1^*/A_1^* of A^*/A_1^* will be denoted by $\{0\}$.

As $\{0\} \times G_B^{*000}$ is a B -invariant subgroup of K of bounded index, $K_B^{000} = \{0\} \times G_B^{*000}$. Hence, $K_B^{000} \cap (A^*/A_1^*) = \{0\}$.

Using this together with the fact that Φ is a B -invariant isomorphism, we conclude that

$$\{0\} = \Phi^{-1}[\{0\}] = \Phi^{-1}[K_B^{000} \cap (A^*/A_1^*)] = H_B^{000} \cap (A^*/A_1^*) = (\widetilde{G}_B^{*000} \cap A^*)/A_1^*.$$

Therefore, $\widetilde{G}_B^{*000} \cap A^* \subseteq A_1^*$, a contradiction.

(ii) We have that $G_B^{*000} = G_B^{*00} = G^*$ is perfect. The implication (\Rightarrow) follows from the implication (\Rightarrow) in point (i).

(\Leftarrow) . The idea is to apply the proof of (\Leftarrow) in (i), noticing that by Beth's definability theorem, the function f considered in the proof of (i) is a B -type-definable subset of $G^* \times (A^*/A_1^*)$ (i.e. its preimage in $G^* \times A^*$ under the map $(g, a) \mapsto (g, a + A_1^*)$ is type-definable over B).

Denote by L the language of \mathcal{G} , and by L_B its expansion by the constants from B . Let $L_1 = L_B \cup \{f_1\}$ and $L_2 = L_B \cup \{f_2\}$, where f_1 and f_2 are two new distinct function symbols. Let $A(x)$ and $G(y)$ be formulas in L defining A and G in \mathcal{G} , respectively. Let $A_1(x)$ be a type (in L_B) defining A_1^* in \mathcal{G}^* . We will use the fact that h is a function definable in \mathcal{G} in the language L_B .

For $i \in \{1, 2\}$ we define a theory T_i in the language L_i as the theory of \mathcal{G} in L_B together with the sentence

$$(\forall x) (G(x) \rightarrow A(f_i(x)))$$

and the following collection of formulas in L_i

$$(\forall x, y) [(G(x) \wedge G(y)) \rightarrow \varphi(h(x, y) - f_i(xy) + f_i(x) + f_i(y))]$$

with $\varphi(z)$ ranging over all formulas from $A_1(z)$. This extra collection of formulas says that in any model M of T_i , the induced 2-cocycle $\bar{h}^M: G(M) \times G(M) \rightarrow A(M)/A_1(M)$ coincides with the function $\bar{f}_i^M(xy) - \bar{f}_i^M(x) - \bar{f}_i^M(y)$, where $\bar{f}_i^M: G(M) \rightarrow A(M)/A_1(M)$ is given by $\bar{f}_i^M(x) = f_i^M(x) + A_1(M)$ (where f_i^M is the interpretation of f_i in M).

It follows easily that whenever M is a model of $T_1 \cup T_2$, then the difference $\bar{f}_1^M - \bar{f}_2^M$ belongs to $\text{Hom}(G(M), A(M)/A_1(M))$, which is trivial because $G(M)$ is perfect and $A(M)/A_1(M)$ is abelian. So $\bar{f}_1^M = \bar{f}_2^M$. Therefore, $T_1 \cup T_2 \models p_1(x, y) \equiv p_2(x, y)$, where $p_i(x, y)$ is the type

$$G(x) \wedge A_1(y - f_i(x))$$

in the language L_i .

Using Beth's theorem (i.e. Fact 1.4), we get a type $p(x, y)$ in L_B such that

$$T_1 \models p_1(x, y) \equiv p(x, y).$$

Suppose for a contradiction that $\bar{h}: G^* \times G^* \rightarrow A^*/A_1^*$ is splitting. By Remark 2.7(iii), this means that $\bar{h}(x, y) = f(x) + f(y) - f(xy)$ for some function $f: G^* \rightarrow A^*/A_1^*$. We can write $f(x) = f'(x) + A_1^*$ for some function $f': G^* \rightarrow A^*$.

Expanding \mathcal{G}^* by f' (prolongated arbitrarily outside of G^*) and treating f' as the interpretation of the function symbol f_1 , \mathcal{G}^* becomes a model of T_1 . We conclude that $f(x) = y + A_1^*$ if and only if $\mathcal{G}^* \models p(x, y)$.

Having that f is type-definable over B , we get that Φ (defined in the proof of (i)) is also type-definable over B . Hence, modifying slightly the rest of the proof of (i), one easily gets $\widetilde{G}_{*B}^{*00} \cap A^* \subseteq A_1^*$, which is a contradiction. \square

Propositions 2.8 and 2.9(ii) yield the following observation.

Corollary 2.10. *Consider the situation described in the first two sentences of Theorem 2.1, and suppose $A_1^* \subseteq \widetilde{G}_{*B}^{*00} \cap A^*$. Assume additionally that G is absolutely connected of finite commutator width. Then the conclusion $\widetilde{G}_{*B}^{*00} \neq \widetilde{G}_{*B}^{*00}$ of Theorem 2.1 implies its assumption (i), i.e. the induced 2-cocycle $\bar{h}: G^{*00} \times G^{*00} \rightarrow A_0/(A_1^* \cap A_0)$ is non-splitting.*

Let us summarize Propositions 2.8 and 2.9(i). Consider the situation described in the first two sentences of Theorem 2.1, and suppose $G^{*000} = G^{*00}$. Assume additionally that $A_1^* \subseteq \widetilde{G}_{*B}^{*00} \cap A^*$ and $\text{Hom}(G^{*000}, A_0/(A_1^* \cap A_0))$ is trivial. Then:

- The conclusion of Theorem 2.1 (i.e. $\widetilde{G}_{*B}^{*000} \neq \widetilde{G}_{*B}^{*00}$) implies that $\widetilde{G}_{*B}^{*00} \cap A^* \not\subseteq A_1^*$, and it is equivalent to this condition under the assumption (ii).
- The assumption (i) of Theorem 2.1 is equivalent to the condition $\widetilde{G}_{*B}^{*000} \cap A^* \not\subseteq A_1^*$.

This leads us to the following question.

Question 2.11. Under the above assumptions, is it true that $\widetilde{G}_{*B}^{*00} \cap A^* \subseteq A_1^*$ if and only if $\widetilde{G}_{*B}^{*000} \cap A^* \subseteq A_1^*$?

The positive answer to Question 2.11 would imply that, under the above hypothesis, the conclusion of Theorem 2.1 implies its assumption (i). This is strongly related to the following question.

Question 2.12. Does there exist data with the properties described in the first sentence of Theorem 2.1 together with $G^{*000} = G^{*00}$, and such that $\widetilde{G}_{*B}^{*000} \cap A^*$ is type-definable but different from $\widetilde{G}_{*B}^{*00} \cap A^*$?

The above question is interesting due to several reasons. If the answer to this question is negative, then the answer to Question 2.11 is positive, so the conclusion of Theorem 2.1 implies the assumption (i) (under the hypothesis described before Question 2.11). If the answer is positive, then putting $A_1^* = \widetilde{G}_{*B}^{*000} \cap A^*$, we get a situation satisfying all the requirements described in the first two sentences of Theorem 2.1 together with the conclusion of this theorem, but the assumption (i) is not satisfied by Proposition 2.9(i). On the other hand, it will be shown in a forthcoming paper (joint with A. Pillay and S. Solecki) that the positive answer would refute a certain difficult conjecture on Borel complexity of the relation of being in the same Lascar strong type.

Now, we will answer Questions 2.11 and 2.12 in two special (but still rather general) situations.

For example, we will show that the answer to Question 2.11 is positive provided that A_1^* is an intersection of B -definable subgroups of finite index which are invariant under the action of G^* . If the action of G on A is trivial, this extra assumption is equivalent to the fact that A_1^* contains A_B^{*0} (the intersection of all B -definable in \mathcal{G}^* , finite index subgroups of A^*). In particular, if the action of G on A is trivial, $G_B^{*000} = G_B^{*00}$ is perfect, and $A_1^* = A_B^{*000} = A_B^{*0}$, we get that the assumption (i) of Theorem 2.1 is equivalent to its conclusion (for (\Rightarrow) we need to assume (ii), for (\Leftarrow) we do not).

In fact, in the next proposition, we consider a more general context than in the above discussion.

Proposition 2.13. (i) *Consider the situation described in the first two sentences of Theorem 2.1, and assume that G is absolutely connected of finite commutator width. Then $\widetilde{G}_B^{*00} \cap A^* \subseteq A_1^*$ if and only if $\widetilde{G}_B^{*000} \cap A^* \subseteq A_1^*$.*
(ii) *Consider the situation described in the first sentence of Theorem 2.1. Let A_1^* be a bounded index subgroup of A^* which is invariant over B , invariant under the action of G^* , and which is an intersection of definable subgroups of finite index. Assume $G_B^{*000} = G_B^{*00}$. Then $\widetilde{G}_B^{*00} \cap A^* \subseteq A_1^*$ if and only if $\widetilde{G}_B^{*000} \cap A^* \subseteq A_1^*$.*

Proof. (i) Only the implication (\Leftarrow) requires an explanation. Assume $\widetilde{G}_B^{*000} \cap A^* \subseteq A_1^*$. By Proposition 2.9(i), this implies that $\bar{h}: G_B^{*000} \times G_B^{*000} \rightarrow A_0/(A_1^* \cap A_0)$ is splitting. Thus, since $G_B^{*000} = G_B^{*00}$, Proposition 2.9(ii) gives us that $\widetilde{G}_B^{*00} \cap A^* \subseteq A_1^*$.

(ii) Once again only the implication (\Leftarrow) requires a proof. We start from the following claim.

Claim. The group A_1^* can be written as $\bigcap_{i \in I} A_i$, where all A_i 's are B -definable (in \mathcal{G}^*) subgroups of A^* of finite index, invariant under the action of G^* and so normal in \widetilde{G}^* .

Proof of the claim. We can write $A_1^* = \bigcap_{i \in I} C_i$, where all C_i 's are definable subgroups of A^* of finite index. Since $A_1^* \leq C_i$, A_1^* is invariant under G^* , and A^*/A_1^* is of bounded size, we get that the orbit of the set C_i under G^* is of bounded size, and so the set-wise stabilizer $\text{Stab}_{G^*}(C_i)$ of C_i in G^* is a definable subgroup of bounded index. Thus, $[G^* : \text{Stab}_{G^*}(C_i)] < \aleph_0$. Define B_i as the intersection of all $g \cdot C_i$ for g ranging over G^* . We conclude that $A_1^* = \bigcap_{i \in I} B_i$, and all B_i 's are definable subgroups of A^* of finite index, invariant under G^* .

Since $A_1^* \leq B_i$, A_1^* is invariant over B , and A^*/A_1^* is of bounded size, we get that the orbit of the definable set B_i under $\text{Aut}(\mathcal{G}^*/B)$ is of bounded size, and so it is finite. Let A_i be the intersection of all $f[B_i]$ for f ranging over $\text{Aut}(\mathcal{G}^*/B)$. We conclude that $A_1^* = \bigcap_{i \in I} A_i$, and all A_i 's are B -definable subgroups of A^* of finite index, invariant under G^* . \square

The quotient \widetilde{G}^*/A_i can and will be freely identified with the extension of G^* by A^*/A_i defined by means of the 2-cocycle induced by h and the action of G^* on A^*/A_i induced by the action of G^* on A^* . Then, $\pi: \widetilde{G}^*/A_i \rightarrow G^*$ denotes the projection on the second coordinate.

We see that $\left(\widetilde{G^*}/A_i\right)_B^{00} = \left(\widetilde{G_B^{*00}} + A_i\right)/A_i$ and $\left(\widetilde{G^*}/A_i\right)_B^{000} = \left(\widetilde{G_B^{*000}} + A_i\right)/A_i$. Since $G_B^{*000} = G_B^{*00}$, we conclude that

$$\pi \left[\left(\widetilde{G^*}/A_i\right)_B^{00} \right] = G_B^{*00} = G_B^{*000} = \pi \left[\left(\widetilde{G^*}/A_i\right)_B^{000} \right].$$

Therefore, $(A^*/A_i) \cdot \left(\widetilde{G^*}/A_i\right)_B^{000} \geq \left(\widetilde{G^*}/A_i\right)_B^{00}$. Using the assumption that A^*/A_i is finite, we conclude that

$$\left[\left(\widetilde{G^*}/A_i\right)_B^{00} : \left(\widetilde{G^*}/A_i\right)_B^{000} \right] < \aleph_0.$$

In virtue of [7, Lemma 3.9], this implies that $\left(\widetilde{G^*}/A_i\right)_B^{00} = \left(\widetilde{G^*}/A_i\right)_B^{000}$. Therefore, $\widetilde{G_B^{*00}} \cap A^* \subseteq \left(\widetilde{G_B^{*000}} \cap A^*\right) + A_i$. Since we have assumed that $\widetilde{G_B^{*000}} \cap A^* \subseteq A_1^* \subseteq A_i$, we conclude that $\widetilde{G_B^{*00}} \cap A^* \subseteq A_i$. As this holds for any $i \in I$, we get $\widetilde{G_B^{*00}} \cap A^* \subseteq A_1^*$. \square

The next, obvious corollary of the above proposition tells us that the answer to Question 2.12 is positive under the additional assumption that G is absolutely connected of finite commutator width or that $\widetilde{G_B^{*000}} \cap A^*$ is an intersection of definable subgroups of finite index in A^* .

Corollary 2.14. *Consider the situation from the first sentence of Theorem 2.1.*

(i) *Assume additionally that G is absolutely connected of finite commutator width. Then, if $\widetilde{G_B^{*000}} \cap A^*$ is type-definable, it must coincide with $\widetilde{G_B^{*00}} \cap A^*$, and so $\widetilde{G_B^{*000}} = \widetilde{G_B^{*00}}$.*

(ii) *Assume additionally that $G_B^{*000} = G_B^{*00}$. Then, if $\widetilde{G_B^{*000}} \cap A^*$ is an intersection of definable subgroups of finite index in A^* , then $\widetilde{G_B^{*000}} \cap A^* = \widetilde{G_B^{*00}} \cap A^*$, and so $\widetilde{G_B^{*000}} = \widetilde{G_B^{*00}}$.*

In Remark 2.5, we have seen that if $G_B^{*000} = G_B^{*00}$, then the quotient $\widetilde{G_B^{*00}}/\widetilde{G_B^{*000}}$ is abelian. Now, we give a more precise description of this quotient.

Proposition 2.15. *Consider the situation from the first sentence of Theorem 2.1, and assume that $G_B^{*000} = G_B^{*00}$. Then $\widetilde{G_B^{*00}}/\widetilde{G_B^{*000}}$ is isomorphic to $\left(\widetilde{G_B^{*00}} \cap A^*\right) / \left(\widetilde{G_B^{*000}} \cap A^*\right)$, and so it is abelian. More precisely, there exists a B -invariant isomorphisms between these groups.*

Proof. For each $g \in G_B^{*000} = G_B^{*00}$ choose $a_g \in A^*$ so that $(a_g, g) \in \widetilde{G_B^{*000}}$. Using the exact sequence (3), one easily gets:

$$\begin{aligned} \widetilde{G_B^{*000}} \cap (A^* \times \{g\}) &= \left(a_g + \left(\widetilde{G_B^{*000}} \cap A^*\right)\right) \times \{g\}, \\ \widetilde{G_B^{*00}} \cap (A^* \times \{g\}) &= \left(a_g + \left(\widetilde{G_B^{*00}} \cap A^*\right)\right) \times \{g\}. \end{aligned}$$

We will show that the formula

$$\Phi \left((a, g) \widetilde{G_B^{*000}} \right) = a - a_g + \left(\widetilde{G_B^{*000}} \cap A^* \right)$$

is a well definition of a B -invariant isomorphism

$$\Phi: \widetilde{G}_B^{*00} / \widetilde{G}_B^{*000} \rightarrow \left(\widetilde{G}_B^{*00} \cap A^* \right) / \left(\widetilde{G}_B^{*000} \cap A^* \right).$$

We check that Φ is well-defined. Consider any $(a_1, g_1), (a_2, g_2) \in \widetilde{G}_B^{*00}$ such that $(a_1, g_1)(a_2, g_2)^{-1} \in \widetilde{G}_B^{*000}$. Our goal is to show that $(a_1 - a_{g_1}) - (a_2 - a_{g_2}) \in \widetilde{G}_B^{*000} \cap A^*$.

Since $(a_{g_1}, g_1), (a_{g_2}, g_2) \in \widetilde{G}_B^{*000} \cap A^*$, we have:

$$\begin{aligned} (a_1 - g_1 g_2^{-1} \cdot a_2 - g_1 \cdot h(g_2^{-1}, g_2) + h(g_1, g_2^{-1}), g_1 g_2^{-1}) &= (a_1, g_1)(a_2, g_2)^{-1} \in \widetilde{G}_B^{*00} \cap A^*, \\ (a_{g_1} - g_1 g_2^{-1} \cdot a_{g_2} - g_1 \cdot h(g_2^{-1}, g_2) + h(g_1, g_2^{-1}), g_1 g_2^{-1}) &= (a_{g_1}, g_1)(a_{g_2}, g_2)^{-1} \in \widetilde{G}_B^{*000} \cap A^*. \end{aligned}$$

This implies

$$(a_1 - a_{g_1}) - g_1 g_2^{-1} \cdot (a_2 - a_{g_2}) \in \widetilde{G}_B^{*000} \cap A^*.$$

On the other hand, by Remark 2.7(ii),

$$g_1 g_2^{-1} \cdot (a_2 - a_{g_2}) - (a_2 - a_{g_2}) \in \widetilde{G}_B^{*000} \cap A^*.$$

So, we conclude that $(a_1 - a_{g_1}) - (a_2 - a_{g_2}) \in \widetilde{G}_B^{*000} \cap A^*$.

It is clear that Φ is invariant over B and that it is a surjection. Hence, it remains to check that it is an injective homomorphism.

First, we check that Φ is a homomorphism. Take any $(a_1, g_1), (a_2, g_2) \in \widetilde{G}_B^{*00}$. Then, $\Phi \left((a_1, g_1)(a_2, g_2) \widetilde{G}_B^{*000} \right) = a_1 + g_1 \cdot a_2 + h(g_1, g_2) - a_{g_1 g_2} + \left(\widetilde{G}_B^{*000} \cap A^* \right)$ and

$$\Phi \left((a_1, g_1) \widetilde{G}_B^{*000} \right) + \Phi \left((a_2, g_2) \widetilde{G}_B^{*000} \right) = a_1 - a_{g_1} + a_2 - a_{g_2} + \left(\widetilde{G}_B^{*000} \cap A^* \right).$$

So, our goal is to show that

$$(a_{g_1} + g_1 \cdot a_2 + h(g_1, g_2) - a_{g_1 g_2}) - (a_2 - a_{g_2}) \in \left(\widetilde{G}_B^{*000} \cap A^* \right).$$

Since $(a_{g_1} + g_1 \cdot a_{g_2} + h(g_1, g_2), g_1 g_2) = (a_{g_1}, g_1)(a_{g_2}, g_2) \in \widetilde{G}_B^{*000}$, we have that $a_{g_1} + g_1 \cdot a_{g_2} + h(g_1, g_2) - a_{g_1 g_2} \in \widetilde{G}_B^{*000} \cap A^*$. Therefore,

$$\begin{aligned} (a_{g_1} + g_1 \cdot a_2 + h(g_1, g_2) - a_{g_1 g_2}) - (a_2 - a_{g_2}) &= (a_{g_1} + g_1 \cdot a_{g_2} + h(g_1, g_2) - a_{g_1 g_2}) \\ &+ g_1 \cdot (a_2 - a_{g_2}) - (a_2 - a_{g_2}) \in g_1 \cdot (a_2 - a_{g_2}) - (a_2 - a_{g_2}) + \left(\widetilde{G}_B^{*000} \cap A^* \right). \end{aligned}$$

We are done, because Remark 2.7(ii) implies that $g_1 \cdot (a_2 - a_{g_2}) - (a_2 - a_{g_2}) \in \widetilde{G}_B^{*000} \cap A^*$.

It remains to show that Φ is injective. Consider any $(a_1, g_1), (a_2, g_2) \in \widetilde{G}_B^{*00}$ such that $(a_1, g_1)(a_2, g_2)^{-1} \notin \widetilde{G}_B^{*000}$. Then,

$$(8) \quad a_1 - g_1 g_2^{-1} \cdot a_2 - g_1 \cdot h(g_2^{-1}, g_2) + h(g_1, g_2^{-1}) - a_{g_1 g_2^{-1}} \notin \widetilde{G}_B^{*000} \cap A^*.$$

Our goal is to show that $(a_1 - a_{g_1}) - (a_2 - a_{g_2}) \notin \widetilde{G}_B^{*000} \cap A^*$. Suppose for a contradiction that this is not the case.

As $(a_{g_1}, g_1), (a_{g_2}, g_2) \in \widetilde{G}_B^{*000}$, we have that

$$(a_{g_1} - g_1 g_2^{-1} \cdot a_{g_2} - g_1 \cdot h(g_2^{-1}, g_2) + h(g_1, g_2^{-1}), g_1 g_2^{-1}) = (a_{g_1}, g_1)(a_{g_2}, g_2)^{-1} \in \widetilde{G}_B^{*000},$$

and so

$$a_{g_1} - g_1 g_2^{-1} \cdot a_{g_2} - g_1 \cdot h(g_2^{-1}, g_2) + h(g_1, g_2^{-1}) - a_{g_1 g_2^{-1}} \in \widetilde{G}_B^{*000} \cap A^*.$$

Using this together with the assumption that $(a_1 - a_{g_1}) - (a_2 - a_{g_2}) \in \widetilde{G}_B^{*000} \cap A^*$ and the fact $g_1 g_2^{-1} \cdot (a_{g_2} - a_2) - (a_{g_2} - a_2) \in \widetilde{G}_B^{*000} \cap A^*$ (which follows from Remark 2.7(ii)), we get the following computation, which contradicts (8):

$$\begin{aligned} & a_1 - g_1 g_2^{-1} \cdot a_2 - g_1 \cdot h(g_2^{-1}, g_2) + h(g_1, g_2^{-1}) - a_{g_1 g_2^{-1}} \\ &= \left(a_{g_1} - g_1 g_2^{-1} \cdot a_{g_2} - g_1 \cdot h(g_2^{-1}, g_2) + h(g_1, g_2^{-1}) - a_{g_1 g_2^{-1}} \right) \\ &+ (a_1 - a_{g_1}) + g_1 g_2^{-1} \cdot (a_{g_2} - a_2) \in \widetilde{G}_B^{*000} \cap A^*. \end{aligned}$$

□

In the examples from [3], and, more generally, for each group G definable in a monster model of an \mathcal{o} -minimal expansion of a real closed field, the quotient G^{00}/G^{000} turns out to be abstractly isomorphic to an abelian, compact Lie group divided by a dense, finitely generated subgroup [4, Remark 5.5].

In the situation from Theorem 2.1, assuming that $G_B^{*000} = G_B^{*00}$ and $A_1^* \subseteq \widetilde{G}_B^{*000} \cap A^*$, and using Proposition 2.15, we get

$$\widetilde{G}_B^{*00}/\widetilde{G}_B^{*000} \cong \left(\left(\widetilde{G}_B^{*00} \cap A^* \right) / A_1^* \right) / \left(\left(\widetilde{G}_B^{*000} \cap A^* \right) / A_1^* \right).$$

Thus, since $\widetilde{G}_B^{*000} \cap A^* \leq A_1^* + A_0$, we easily conclude that $\widetilde{G}_B^{*00}/\widetilde{G}_B^{*000}$ is isomorphic to the quotient of an abelian, compact group by a finitely generated subgroup. A question arises if this finitely generated subgroup is dense. Proposition 2.13 yields the following observation.

Proposition 2.16. (i) *Consider the situation described in the first two sentences of Theorem 2.1, and assume that G is absolutely connected of finite commutator width. Then $\left(\widetilde{G}_B^{*000} \cap A^* \right) / A_1^*$ is dense in $\left(\widetilde{G}_B^{*00} \cap A^* \right) / A_1^*$.*

(ii) *Consider the situation described in the first sentence of Theorem 2.1. Let A_1^* be a bounded index subgroup of A^* which is invariant over B , invariant under the action of G^* , and which is an intersection of definable subgroups of finite index. Assume $G_B^{*000} = G_B^{*00}$. Then $\left(\widetilde{G}_B^{*000} \cap A^* \right) / A_1^*$ is dense in $\left(\widetilde{G}_B^{*00} \cap A^* \right) / A_1^*$.*

Proof. (i) Let $\rho: A^* \rightarrow A^*/A_1^*$ be the quotient map, and let A_2^* be the preimage under ρ of the closure of $\left(\widetilde{G}_B^{*000} \cap A^* \right) / A_1^*$. We see that A_2^* is a bounded index, B -type-definable subgroup of A^* invariant under the action of G^* . Moreover, $\widetilde{G}_B^{*000} \cap A^* \subseteq A_2^*$. By Proposition 2.13(i) applied to A_2^* (instead of A_1^*), we get that $\widetilde{G}_B^{*00} \cap A^* \subseteq A_2^*$, which implies that $\left(\widetilde{G}_B^{*000} \cap A^* \right) / A_1^*$ is dense in $\left(\widetilde{G}_B^{*00} \cap A^* \right) / A_1^*$.

(ii) By the claim from the proof of Proposition 2.13, we know that $A_1^* = \bigcap_{i \in I} A_i$, where all A_i 's are B -definable subgroups of A^* of finite index, invariant under G^* . We can assume that the family $\{A_i : i \in I\}$ is closed under finite intersections. Take the notation from the proof of (i). Since A_2^* is B -type-definable and contains A_1^* , we get that $A_2^* = \bigcap_{i \in I} (A_i + A_2^*)$ and each $A_i + A_2^*$ is a B -definable subgroup of A^* of finite index,

invariant under G^* . Moreover, $\widetilde{G}_B^{*000} \cap A^* \subseteq A_2^*$. Thus, applying Proposition 2.13(ii), we get that $\widetilde{G}_B^{*00} \cap A^* \subseteq A_2^*$, which is enough. \square

3. EXTENSIONS

In this section, we make a few observations telling us that in some situations in which Theorem 2.1 or Corollary 2.2 can be applied to a group G , it can also be applied to certain extensions of G (providing new examples in Section 4).

Suppose G acts on an abelian group A , $h: G \times G \rightarrow A$ is a 2-cocycle and $f: H \rightarrow G$ is an epimorphism. Then f induces an action of H on A so that the composition $h' := h \circ (f, f): H \times H \rightarrow A$ becomes a 2-cocycle on H . By using the classical idea of the “Inflation-Restriction Sequence” [15, Theorem 4.1.20], we obtain the following proposition.

Proposition 3.1. *Under the notation of the previous paragraph, suppose $z: G \rightarrow H$ is a section of f with $z(e_G) = e_H$. The 2-cocycle $h' = h \circ (f, f): H \times H \rightarrow A$ is splitting if and only if there exists a homomorphism $g: \ker(f) \rightarrow A$ which is G -invariant, that is for $x' \in H$ and $y' \in \ker(f)$ we have $g(x'y'x'^{-1}) = f(x') \cdot g(y')$, and such that h is cohomologous with the 2-cocycle h'' defined by*

$$h''(x, y) = g(z(xy)z(y)^{-1}z(x)^{-1})$$

for $x, y \in G$.

In particular, if $\text{Hom}(\ker(f), A)$ is trivial, then h is non-splitting if and only if h' is non-splitting.

Proof. (\Rightarrow). There exists a function $g: H \rightarrow A$ such that for $x', y' \in H$,

$$(\diamond) \quad h(f(x'), f(y')) = g(x') + f(x') \cdot g(y') - g(x'y').$$

If $x' \in \ker(f)$ or $y' \in \ker(f)$, then $g(x'y') = g(x') + f(x') \cdot g(y')$ (because $h(e_G, -) = h(-, e_G) = 0$), so $g|_{\ker(f)} \in \text{Hom}(\ker(f), A)$. Moreover, taking $x' \in H$ and $y' \in \ker(f)$, we have $x'y'x'^{-1} \in \ker(f)$, so $g(x'y') = g(x'y'x'^{-1}x') = g(x'y'x'^{-1}) + g(x')$, and therefore $g(x'y'x'^{-1}) = g(x'y') - g(x') = f(x') \cdot g(y')$. This means that $g|_{\ker(f)}$ is G -invariant.

Denote $\tilde{z} = g \circ z: G \rightarrow A$. For $x, y \in G$ let $x' = z(x)$ and $y' = z(y)$. We have

$$\begin{aligned} h(x, y) = h(f(x'), f(y')) &= g(z(x)) + x \cdot g(z(y)) - g(z(x)z(y)) \\ &= [g(z(x)) + x \cdot g(z(y)) - g(z(xy))] + g(z(xy)) - g(z(x)z(y)) \\ &= [\tilde{z}(x) + x \cdot \tilde{z}(y) - \tilde{z}(xy)] + g(z(xy)z(y)^{-1}z(x)^{-1}), \end{aligned}$$

because $z(xy)z(y)^{-1}z(x)^{-1} \in \ker(f)$.

(\Leftarrow). If h is cohomologous with h'' , then $h' = h \circ (f, f)$ is cohomologous with $h''' = h'' \circ (f, f)$. For $x', y' \in H$ we have $h'''(x', y') = g(z(f(x'y'))z(f(y'))^{-1}z(f(x'))^{-1}) = g((x'y'z(f(x'y'))^{-1})^{-1}x'(y'z(f(y'))^{-1})x'^{-1}(x'z(f(x'))^{-1})) = F(x') + f(x') \cdot F(y') - F(x'y')$, where $F: H \rightarrow A$ is defined as $F(t) = g(tz(f(t))^{-1})$. Hence, both h''' and h' are splitting. \square

Corollary 3.2. *Consider the situation described in the first part of Theorem 2.1. Suppose that*

- (1) G is absolutely connected,
- (2) $f: H \twoheadrightarrow G$ is an epimorphism.

Regard H and f as objects \emptyset -definable in some first order expansion \mathcal{H} of \mathcal{G} . Let $\mathcal{H}^* \succ \mathcal{H}$ be a big enough monster model. Assume additionally that $\text{Hom}(\ker(f^*), A_0/(A_1^* \cap A_0))$ is trivial, where f^* is the interpretation of f in \mathcal{H}^* . Then the 2-cocycle $\overline{h'}$ defined as

$$\overline{h'} = \overline{h} \circ (f^*, f^*): H^{*00}_B \times H^{*00}_B \rightarrow A_0/(A_1^* \cap A_0),$$

where $\overline{h}: G^* \times G^* \rightarrow A_0/(A_1^* \cap A_0)$ is the induced 2-cocycle, is non-splitting. Therefore, by Theorem 2.1, $\widetilde{H^{*00}_B} \neq \widetilde{H^{*00}_B}$, where \widetilde{H} is the extension of H by A corresponding to the 2-cocycle $h' := h \circ (f, f)$.

Proof. Choose a monster model $\mathcal{H}^* \succ \mathcal{H}$ so that the interpretation of \mathcal{G} in it is a monster model of $\text{Th}(\mathcal{G})$ being an elementary extension of \mathcal{G}^* ; we may assume that this interpretation coincides with \mathcal{G}^* .

Since G is absolutely connected, we have that G^{*00}_B (computed in the sense of \mathcal{H}^*) equals G^* , and so $f^*[H^{*00}_B] = G^*$. If $\overline{h'}$ is splitting, then by Proposition 3.1 and the triviality of $\text{Hom}(\ker(f^*), A_0/(A_1^* \cap A_0))$, the 2-cocycle $\overline{h}: G^* \times G^* \rightarrow A_0/(A_1^* \cap A_0)$ is splitting, which contradicts the assumption (i) of Theorem 2.1. \square

If in Corollary 3.2 one starts from the situation described in the first part of Corollary 2.2 (instead of Theorem 2.1), then the extra assumption that $\text{Hom}(\ker(f^*), A_0/(A_1^* \cap A_0))$ is trivial means that $\text{Hom}(\ker(f^*), \mathbb{Z}^n)$ is trivial, which is always satisfied for example when $\ker(f^*)$ is divisible or finite.

Corollary 3.2 is a general recipe for obtaining new examples of extensions to which Theorem 2.1 can be applied. The next remark is a variant of this, where we assume that H is a product extension of G , but the assumptions that ‘ G is absolutely connected’ and ‘ $\text{Hom}(\ker(f^*), A_0/(A_1^* \cap A_0))$ is trivial’ are dropped.

Remark 3.3. Consider the situation described in the first part of Theorem 2.1. Let $H = K \times G$, where K is an arbitrary group. Let \mathcal{H} be the expansion of \mathcal{G} obtained by adding a new sort, consisting of the pure group structure H , and the projection $f: H \rightarrow G$ on the second coordinate; $\mathcal{H}^* \succ \mathcal{H}$ denotes a big enough monster model.

Then the 2-cocycle $\overline{h'}$ defined as

$$\overline{h'} = \overline{h} \circ (f^*, f^*): H^{*00}_B \times H^{*00}_B \rightarrow A_0/(A_1^* \cap A_0),$$

is non-splitting. Therefore, by Theorem 2.1, $\widetilde{H^{*00}_B} \neq \widetilde{H^{*00}_B}$, where \widetilde{H} is an extension of H corresponding to the 2-cocycle $h' = h \circ (f, f)$.

Proof. Let \mathcal{H}_1 be the expansion of \mathcal{G} by the additional sort for the pure group structure K ; then the group H and the projection f are definable in \mathcal{H}_1 . Take a monster model $\mathcal{H}_1^* \succ \mathcal{H}_1$ so big that the interpretation of \mathcal{H} in \mathcal{H}_1^* (which we denote by \mathcal{H}^*) is a monster model of $\text{Th}(\mathcal{H})$ and the interpretation of \mathcal{G} in \mathcal{H}_1^* is a monster model of $\text{Th}(\mathcal{G})$ being an elementary extension of \mathcal{G}^* ; we may assume that the interpretation of \mathcal{G} in \mathcal{H}_1^* coincides with \mathcal{G}^* . Let K^* and H^* be the interpretations of K and H in \mathcal{H}_1^* .

Then $H^* = K^* \times G^*$. Since $h'((k_1, g_1), (k_2, g_2)) = h(g_1, g_2)$ and $h: G^{*00}_B \times G^{*00}_B \rightarrow A_0/(A_1^* \cap A_0)$ is non-splitting (by assumption), in order to show that $\overline{h'}: H^{*00}_B \times H^{*00}_B \rightarrow A_0/(A_1^* \cap A_0)$ is non-splitting, it is enough to notice G^{*00}_B computed in the sense of \mathcal{H}_1^* coincides with G^{*00}_B computed in the sense of \mathcal{G}^* . But the last equality follows immediately from an easy observation that the structure induced on \mathcal{G}^* from \mathcal{H}_1^* coincides with the original structure on \mathcal{G}^* . \square

Part (2) of the next proposition says that any strongly non-splitting extension of an absolutely connected group by a finite abelian group is also absolutely connected. This is a generalization of [5, Proposition 2.13], where we considered extensions by $\mathbb{Z}/p\mathbb{Z}$ for prime numbers p .

Proposition 3.4. *Let*

$$1 \hookrightarrow A' \hookrightarrow H \xrightarrow{f} G \twoheadrightarrow 1$$

be an extension of a group G by a finite abelian group A' , defined by means of a 2-cocycle $h_f: G \times G \rightarrow A'$. Assume h_f is strongly non-splitting in the following way: for every proper subgroup $A'' \subsetneq A'$ normal in H (equivalently, invariant under the action of G) the induced 2-cocycle $\overline{h_f}: G \times G \rightarrow A'/A''$ is non-splitting. Let $k = |A'|$.

- (1) *The group G has no proper subgroups of finite index if and only if H has no proper subgroups of finite index.*
- (2) *The group G is absolutely connected if and only if H is absolutely connected.*

Proof. (1) The implication (\Leftarrow) is obvious. We prove (\Rightarrow) . Suppose $H_1 \leq H$ is a subgroup of finite index. We may assume that H_1 is a normal subgroup of H . Then $f[H_1] = G$, so $H = H_1 \cdot A'$. We prove that $H = H_1$. Suppose for a contradiction that $A'' = H_1 \cap A'$ is a proper subgroup of A' . Consider the induced epimorphism $\overline{f}: H/A'' \rightarrow G$. Since $A' = \ker(f)$, $\overline{f}|_{H_1/A''}$ is an isomorphism onto G . Hence, the inverse isomorphism $(\overline{f}|_{H_1/A''})^{-1}: G \rightarrow H_1/A''$ is a section of \overline{f} . Therefore, the exact sequence

$$1 \hookrightarrow A'/A'' \hookrightarrow H/A'' \xrightarrow{f} G \twoheadrightarrow 1$$

is splitting, and so the 2-cocycle $\overline{h_f}: G \times G \rightarrow A'/A''$ is also splitting, a contradiction.

(2) By [5, Proposition 2.10(4)], it is enough to prove that H has no proper subgroups of finite index, which follows by (1). \square

The next corollary follows from Proposition 3.1 and Proposition 3.4.

Corollary 3.5. *Consider the situation described in the first part of Corollary 2.2. Suppose that*

- (1) *G is absolutely connected,*
- (2) *$f: H \twoheadrightarrow G$ is an epimorphism with finite and abelian $\ker(f)$, and a 2-cocycle $h_f: G \times G \rightarrow \ker(f)$ corresponding to the extension f is strongly non-splitting in the sense of Proposition 3.4.*

Regard H and f as objects \emptyset -definable in some first order expansion \mathcal{H} of \mathcal{G} . Let $\mathcal{H}^ \succ \mathcal{H}$ be a big enough monster model. Put $h' = h \circ (f, f): H \times H \rightarrow A_0$. Then H is absolutely connected, and the 2-cocycle h' is non-splitting. Therefore, the assumptions of the first part of Corollary 2.2 are satisfied, so $\widetilde{H}^*_{B^{000}} \neq \widetilde{H}^*_{B^{00}}$, where \widetilde{H} is the extension of H corresponding to the 2-cocycle h' .*

Example 4.13 from Section 4 shows that even if in Corollary 3.5 we start from a situation in which all the assumptions of both parts of Corollary 2.2 are satisfied, it may happen that $\widetilde{H}^*_{B^{00}} \neq \widetilde{H}^*$.

4. CENTRAL EXTENSIONS OF $\mathrm{SL}_2(k)$

The aim of this section is to find new (concrete) examples of groups G together with a 2-cocycle $h: G \times G \rightarrow \mathbb{Z}$ for which Theorem 2.1 or Corollary 2.2 can be applied, yielding the groups \widetilde{G}^* satisfying $\widetilde{G}^{*000}_B \neq \widetilde{G}^{*00}_B$. This goal is achieved in Examples 4.9 – 4.13.

We mainly concentrate on the case when $G = \mathrm{SL}_2(k)$ for various infinite fields k , but we hope that Theorem 2.1 can also be applied to higher dimensional symplectic groups $\mathrm{Sp}_{2n}(k)$ (note that $\mathrm{Sp}_2(k) = \mathrm{SL}_2(k)$).

The main problem is to find 2-cocycles with finite image. We will use the Steinberg description of the abstract universal central extension of $\mathrm{SL}_2(k)$. Every central extension of $\mathrm{SL}_2(k)$ by an abelian group A corresponds to a 2-cocycle $\mathrm{SL}_2(k) \times \mathrm{SL}_2(k) \rightarrow A$. However, in this case (more generally, in the case of *split algebraic groups* or *Chevalley groups*), one can use a more convenient approach via *Steinberg symbols* [17, Section 7], which are certain mappings $c: k^\times \times k^\times \rightarrow A$.

For example, the topological universal cover $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ of $\mathrm{SL}_2(\mathbb{R})$ can be described by a symbol defined by [9, p. 51], [10, 10.4]: for $x, y \in \mathbb{R}^\times$

$$(*) \quad c(x, y) = \begin{cases} 1 & \text{if } x < 0 \text{ and } y < 0 \\ 0 & \text{otherwise} \end{cases}.$$

Definition 4.1. [17, p. 74] A central extension $\pi: \widetilde{G} \rightarrow G$ is called *universal* if for any central extension $\pi': G' \rightarrow G$ there exists a unique homomorphism $f: \widetilde{G} \rightarrow G'$ such that $\pi' \circ f = \pi$, that is the following diagram commutes

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\pi} & G \\ f \downarrow & \nearrow \pi' & \\ G' & & \end{array}$$

It is known that any perfect group possesses a universal central extension [17, p. 75], which is unique up to isomorphism over G [17, p. 74].

Suppose k is an arbitrary infinite field, and let k^\times be the multiplicative group of k . Following [14], we describe the universal central extension of $\mathrm{SL}_2(k)$ (see also [17, Sections 6, 7]). For $x \in k$ we define the following matrices:

$$u_{12}(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad u_{21}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

Furthermore, for $x \in k^\times$ and $i, j \in \{1, 2\}$, $i \neq j$, define

$$(O) \quad w_{ij}(x) = u_{ij}(x) u_{ji}(-x^{-1}) u_{ij}(x), \quad h_{ij}(x) = w_{ij}(x) w_{ij}(-1).$$

We have

$$\begin{aligned} w_{12}(x) &= \begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix}, & w_{21}(x) &= \begin{pmatrix} 0 & -x^{-1} \\ x & 0 \end{pmatrix}, \\ h_{12}(x) &= \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, & h_{21}(x) &= \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}. \end{aligned}$$

The group $\mathrm{SL}_2(k)$ is generated by the elements $u_{12}(x), u_{21}(x)$ for $x \in k$. Furthermore, one can describe the presentations of $\mathrm{SL}_2(k)$, its universal central extension and the kernel in terms of generators and relations, which we present below.

Consider the following relations among the elements $u_{ij}(x), w_{ij}(x)$ and $h_{ij}(t)$: for $x, y \in k$ and $t \in k^\times$

- (A) $u_{ij}(x + y) = u_{ij}(x) u_{ij}(y),$
- (B') $w_{ij}(t) u_{ij}(x) w_{ij}(t)^{-1} = u_{ji}(-t^{-2}x),$
- (C) $h_{ij}(xy) = h_{ij}(x) h_{ij}(y).$

The group $\mathrm{SL}_2(k)$ can be presented as

$$\langle u_{ij}(x), w_{ij}(s), h_{ij}(t) \mid (\text{O}), (\text{A}), (\text{B}'), (\text{C}) \rangle_{\{i,j\}=\{1,2\}, x \in k, s \in k^\times, t \in k^\times}.$$

Let $\mathrm{St}_2(k)$ (*the Steinberg group*) be a group with the following presentation

$$\langle u_{ij}(x), w_{ij}(s), h_{ij}(t) \mid (\text{O}), (\text{A}), (\text{B}') \rangle_{\{i,j\}=\{1,2\}, x \in k, s \in k^\times, t \in k^\times}.$$

Then $\mathrm{St}_2(k)$ with the natural mapping π is the universal central extension of $\mathrm{SL}_2(k)$ (recall that k is an infinite field) [17, Theorem 10, p. 78]:

$$(**) \quad 1 \longrightarrow \ker(\pi) \longrightarrow \mathrm{St}_2(k) \xrightarrow{\pi} \mathrm{SL}_2(k) \twoheadrightarrow 1.$$

The kernel $\ker(\pi)$ is denoted by $\mathrm{K}_2^{\mathrm{sym}}(k)$. Define for $x, y \in k^\times$ and $i, j \in \{1, 2\}$, $i \neq j$,

$$c_{ij}(x, y) = h_{ij}(x) h_{ij}(y) h_{ij}(xy)^{-1}.$$

Then one can prove that $c_{12}(x, y) = c_{21}(y, x)^{-1}$, $\mathrm{K}_2^{\mathrm{sym}}(k)$ is the center of $\mathrm{St}_2(k)$ and it is generated by the elements $c_{12}(x, y)$. In fact, by the result of Matsumoto-Moore [9, Prop. 5.5, Thm. 5.10, Cor. 5.11], [17, Theorem 12, p. 86] the elements $c_{12}(x, y)$ satisfy those and only those relations which are implied by the relations (S1), (S2), and (S3) written below. So, $\mathrm{K}_2^{\mathrm{sym}}(k)$ can be presented abstractly as

$$\langle c(x, y) \mid (\text{S1}), (\text{S2}), (\text{S3}) \rangle_{x, y \in k^\times},$$

where

- (S1) $c(x, y) c(xy, z) = c(x, yz) c(y, z),$
- (S2) $c(1, 1) = 1, \quad c(x, y) = c(x^{-1}, y^{-1}),$
- (S3) $c(x, y) = c(x, (1 - x)y)$ for $x \neq 1,$

and we can freely identify $c_{12}(x, y)$ with $c(x, y)$. Since $\mathrm{K}_2^{\mathrm{sym}}(k)$ is abelian, from now on, we will use the additive notation in $\mathrm{K}_2^{\mathrm{sym}}(k)$.

Note that c may be regarded as a mapping $c: k^\times \times k^\times \rightarrow \mathrm{K}_2^{\mathrm{sym}}(k)$. More generally, for an arbitrary abelian group A , every mapping $c: k^\times \times k^\times \rightarrow A$ satisfying (S1), (S2) and (S3) is called a *symplectic Steinberg symbol*. For example, the mapping $(*)$ is such.

An important consequence of the above theory is the existence of a one-to-one correspondence between symplectic Steinberg symbols on k with values in A and central extensions of $\mathrm{SL}_2(k)$ by A up to equivalence (see for example [11, Section 11]).

More precisely, suppose $A \hookrightarrow G' \xrightarrow{\pi'} \mathrm{SL}_2(k)$ is a central extension of $\mathrm{SL}_2(k)$. Then, since $\mathrm{St}_2(k)$ is the universal central extension of $\mathrm{SL}_2(k)$, there is a unique mapping f such that the following diagram commutes

$$\begin{array}{ccccc} \mathrm{K}_2^{\mathrm{sym}}(k) & \longrightarrow & \mathrm{St}_2(k) & \xrightarrow{\pi} & \mathrm{SL}_2(k) \\ \downarrow f|_{\mathrm{K}_2^{\mathrm{sym}}(k)} & & \downarrow f & & \downarrow \mathrm{id} \\ A & \longrightarrow & G' & \xrightarrow{\pi'} & \mathrm{SL}_2(k). \end{array}$$

Hence, the composition $c' = f \circ c$ is a symplectic Steinberg symbol. Conversely, by the Matsumoto-Moore theorem, every symplectic Steinberg symbol $c': k^\times \times k^\times \rightarrow A$ is induced by a unique homomorphism $f_{c'}: K_2^{\text{sym}}(k) \rightarrow A$ mapping $c(x, y)$ to $c'(x, y)$ for each $x, y \in k^\times$, and for $N < K_2^{\text{sym}}(k) \times A$ defined as

$$N = \{(x, -f_{c'}(x)) : x \in K_2^{\text{sym}}(k)\}$$

we obtain the following central extension of $\text{SL}_2(k)$

$$(***) \quad 1 \hookrightarrow A \hookrightarrow (\text{St}_2(k) \times A) / N \xrightarrow{\pi'} \text{SL}_2(k) \twoheadrightarrow 1.$$

Notice also that if $H: \text{SL}_2(k) \times \text{SL}_2(k) \rightarrow K_2^{\text{sym}}(k)$ is any 2-cocycle defining the extension $(**)$ (up to equivalence) and $c': k^\times \times k^\times \rightarrow A$ is a symplectic Steinberg symbol, then the homomorphism $f_{c'}: K_2^{\text{sym}}(k) \rightarrow A$ defined above induces a 2-cocycle $H_{c'}: \text{SL}_2(k) \times \text{SL}_2(k) \rightarrow A$ by putting $H_{c'}(a, a') = f_{c'}(H(a, a'))$. Then, the central extension of $\text{SL}_2(k)$ by A defined by means of $H_{c'}$ turns out to be equivalent to the extension $(***)$.

Let us collect some of the facts discussed above as a corollary so that we could easily refer to them later.

Corollary 4.2. *Let $c': k^\times \times k^\times \rightarrow A$ be a symplectic Steinberg symbol. Then there exists a unique homomorphism $f_{c'}: K_2^{\text{sym}}(k) \rightarrow A$ satisfying $f_{c'}(c(x, y)) = c'(x, y)$ for all $x, y \in k^\times$. If $H: \text{SL}_2(k) \times \text{SL}_2(k) \rightarrow K_2^{\text{sym}}(k)$ is a 2-cocycle defining the extension $(**)$, then the formula $H_{c'}(a, a') = f_{c'}(H(a, a'))$ defines a 2-cocycle $H_{c'}: \text{SL}_2(k) \times \text{SL}_2(k) \rightarrow A$.*

In order to apply Corollary 2.2, we need to find non-splitting 2-cocycles on $\text{SL}_2(k)$ with values in \mathbb{Z}^n and with finite image. We will define such 2-cocycles using symplectic Steinberg symbols. To do this, we need to calculate a 2-cocycle corresponding to the universal central extension $(**)$ in terms of Steinberg symbols. An appropriate formula has been given by Matsumoto in [9, 5.12(a)]. However, for the sake of completeness, we also present the relevant result in Proposition 4.3 below.

First, we have to define a section

$$S: \text{SL}_2(k) \rightarrow \text{St}_2(k)$$

of π from $(**)$. Let B be the subgroup of $\text{SL}_2(k)$ consisting of the upper triangular matrices. By the classical Bruhat decomposition, we have that $\text{SL}_2(k)$ is the disjoint union

$$B \cup B \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B.$$

Now, we are ready to define S . Suppose $a \in \text{SL}_2(k)$.

¹⁰ If $a \in B$, then there are unique $x \in k^\times$ and $y \in k$ such that

$$a = h_{12}(x) u_{12}(y) = \begin{pmatrix} x & xy \\ 0 & \frac{1}{x} \end{pmatrix}.$$

Define $S(a) = h_{12}(x) u_{12}(y)$.

²⁰ If $a \in \text{SL}_2(k) \setminus B$, then there are unique $x, z \in k$ and $y \in k^\times$ such that

$$a = u_{12}(x) w_{12}(y) u_{12}(z) = \begin{pmatrix} -\frac{x}{y} & y - \frac{xz}{y} \\ -\frac{1}{y} & -\frac{z}{y} \end{pmatrix}.$$

Define $S(a) = u_{12}(x) w_{12}(y) u_{12}(z)$.

Proposition 4.3. *Let $H: \text{SL}_2(k) \times \text{SL}_2(k) \rightarrow K_2^{\text{sym}}(k)$ be the 2-cocycle corresponding to the section S , that is $H(a, a') = S(a) S(a') S(aa')^{-1}$. Then $H(a, a') =$*

$$\left\{ \begin{array}{ll} c_{12}(x, x') & : a = h_{12}(x) u_{12}(y), a' = h_{12}(x') u_{12}(y') \\ c_{12}(x, y') & : a = h_{12}(x) u_{12}(y), a' = u_{12}(x') w_{12}(y') u_{12}(z') \\ c_{12}(x', -\frac{y}{x'}) & : a = u_{12}(x) w_{12}(y) u_{12}(z), a' = h_{12}(x') u_{12}(y') \\ -c_{12}(-\frac{y}{y'}, -y') & : a, a' \in \text{SL}_2(k) \setminus B \text{ and } z + x' = 0 \\ c_{12}(-\frac{y}{y'}, \frac{y'^2}{z+x'}) - c_{12}(-\frac{y}{y'}, -y') & : a, a' \in \text{SL}_2(k) \setminus B \text{ and } z + x' \neq 0 \end{array} \right. ,$$

where in the last two cases $a = u_{12}(x) w_{12}(y) u_{12}(z)$ and $a' = u_{12}(x') w_{12}(y') u_{12}(z')$.

Proof. We will use several relations which are consequences of the relations (O), (A) and (B') (see [17, Lemma 37]), namely:

- (1) $h_{ij}(t) u_{ij}(x) = u_{ij}(t^2 x) h_{ij}(t)$,
- (2) $h_{ij}(t) u_{ji}(x) = u_{ji}(t^{-2} x) h_{ij}(t)$.

For simplicity denote $h = h_{12}$, $u = u_{12}$, $w = w_{12}$ and $c = c_{12}$.

Case 1⁰. Suppose $a = h(x) u(y)$ and $a' = h(x') u(y')$. Then $aa' = \begin{pmatrix} xx' & xx'y' + \frac{xy}{x'} \\ 0 & \frac{1}{xx'} \end{pmatrix} = h(xx') u(\frac{y}{x'^2} + y')$.

Hence, using the relations (O), (A) and (1), we get

$$\begin{aligned} H(a, a') &= h(x) u(y) h(x') u(y') \left(h(xx') u\left(\frac{y}{x'^2} + y'\right) \right)^{-1} \\ &= h(x) u(y) \left(h(x') u\left(-\frac{y}{x'^2}\right) \right) h(xx')^{-1} \\ &= h(x) h(x') h(xx')^{-1} = c(x, x'). \end{aligned}$$

Case 2⁰. Suppose $a = h(x) u(y)$ and $a' = u(x') w(y') u(z')$. Then $aa' = \begin{pmatrix} -\frac{x}{y'}(y + x') & -\frac{x(x'z' - y'^2 + yz')}{y'} \\ -\frac{1}{xy'} & -\frac{z'}{xy'} \end{pmatrix} = u(x^2(y + x')) w(xy') u(z')$. Hence, using the relations (O), (A) and (1), we get

$$\begin{aligned} H(a, a') &= h(x) u(y + x') w(y') w(xy')^{-1} u(-x^2(y + x')) \\ &= h(x) u(y + x') h(y') h(xy')^{-1} u(-x^2(y + x')) \\ &= h(x) h(y') u\left(\frac{y + x'}{y'^2}\right) h(xy')^{-1} u(-x^2(y + x')) \\ &= h(x) h(y') h(xy')^{-1} = c(x, y'). \end{aligned}$$

Case 3⁰. Suppose $a = u(x) w(y) u(z)$ and $a' = h(x') u(y')$. Then $aa' = \begin{pmatrix} -\frac{xx'}{y} & \frac{y^2 - x(z + x'^2 y')}{y x'} \\ -\frac{x'}{y} & -\frac{y x'^2 y'}{y x'} \end{pmatrix} = u(x) w\left(\frac{y}{x'}\right) u\left(\frac{z}{x'^2} + y'\right)$. Hence, using the relations (O), (A)

and (1), we get

$$\begin{aligned}
H(a, a') &= u(x) w(y) \left(u(z) h(x') u\left(-\frac{z}{x'^2}\right) \right) w\left(\frac{y}{x'}\right)^{-1} u(-x) \\
&= u(x) w(y) h(x') w\left(-\frac{y}{x'}\right) u(-x) \\
&= (u(x) w(y)) \left(h(x') h\left(-\frac{y}{x'}\right) h(-y)^{-1} \right) h(-y) w(-1)^{-1} u(-x) \\
&= (u(x) w(y)) c\left(x', -\frac{y}{x'}\right) (u(x) w(y))^{-1} = c\left(x', -\frac{y}{x'}\right),
\end{aligned}$$

since $c\left(x', -\frac{y}{x'}\right)$ is central in $\text{St}_2(k)$.

Case 4⁰. Suppose $a = u(x) w(y) u(z)$, $a' = u(x') w(y') u(z')$ and $z + x' = 0$. Then $aa' = \begin{pmatrix} -\frac{y}{y'} & -\frac{xy'^2+y^2z'}{yy'} \\ 0 & -\frac{y}{y'} \end{pmatrix} = h\left(-\frac{y}{y'}\right) u\left(\frac{y'^2}{y^2}x + z'\right)$. Hence, using the relations (O), (A) and (1), we get

$$\begin{aligned}
H(a, a') &= u(x) w(y) w(y') \left(u\left(-\frac{y'^2}{y^2}x\right) h\left(-\frac{y}{y'}\right)^{-1} \right) \\
&= u(x) \left(w(y) w(-y')^{-1} h\left(-\frac{y}{y'}\right)^{-1} \right) u(x)^{-1} \\
&= u(x) c\left(-\frac{y}{y'}, -y'\right)^{-1} u(x)^{-1} = -c\left(-\frac{y}{y'}, -y'\right).
\end{aligned}$$

Case 5⁰. Suppose $a = u(x) w(y) u(z)$, $a' = u(x') w(y') u(z')$ and $z + x' \neq 0$. Then $aa' = \begin{pmatrix} \frac{x(z+x')-y^2}{yy'} & \frac{x(zz'+x'z'-y'^2)-z'y^2}{yy'} \\ \frac{z+x'}{yy'} & \frac{z'(z+x')-y'^2}{yy'} \end{pmatrix} = u(x'') w(y'') u(z'')$, where $x'' = x - \frac{y^2}{z+x'}$, $y'' = -\frac{yy'}{z+x'}$ and $z'' = z' - \frac{y'^2}{z+x'}$. Denote $t = z + x'$. Using the relations (O), (A), (B') and (2), we get

$$\begin{aligned}
H(a, a') &= u(x) w(y) u(t) w(y') u(z' - z'') w(-y'') u(-x'') \\
&= u(x) \left(w(y) u(t) w(y') u\left(\frac{y'^2}{t}\right) w\left(\frac{yy'}{t}\right) u\left(\frac{y^2}{t}\right) \right) u(x)^{-1} \\
&= u(x) \left(h(y) [w(1)u(t)w(1)^{-1}] h(-y')^{-1} u\left(\frac{y'^2}{t}\right) w\left(\frac{yy'}{t}\right) u\left(\frac{y^2}{t}\right) \right) u(x)^{-1} \\
&= u(x) (h(y) u_{21}(-t) h(-y')^{-1}) \left(u\left(\frac{y'^2}{t}\right) w\left(\frac{yy'}{t}\right) u\left(\frac{y^2}{t}\right) \right) u(x)^{-1} \\
&= u(x) \left(h(y) h(-y')^{-1} u_{21}\left(-\frac{t}{y'^2}\right) \right) \left(u\left(\frac{y'^2}{t}\right) w\left(\frac{yy'}{t}\right) u\left(\frac{y^2}{t}\right) \right) u(x)^{-1} \\
&= u(x) \left(h(y) h(-y')^{-1} h\left(-\frac{y}{y'}\right)^{-1} \right) \Delta u(x)^{-1} \\
&= c\left(-\frac{y}{y'}, -y'\right)^{-1} u(x) \Delta u(x)^{-1},
\end{aligned}$$

where, by the relations (O), (A) and (1), we have

$$\begin{aligned}
\Delta &= h\left(-\frac{y}{y'}\right) u_{21}\left(-\frac{t}{y'^2}\right) u\left(\frac{y'^2}{t}\right) w\left(\frac{yy'}{t}\right) u\left(\frac{y^2}{t}\right) \\
&= h\left(-\frac{y}{y'}\right) \left[u\left(-\frac{y'^2}{t}\right) u\left(\frac{y'^2}{t}\right) \right] u_{21}\left(-\frac{t}{y'^2}\right) u\left(\frac{y'^2}{t}\right) w\left(\frac{yy'}{t}\right) u\left(\frac{y^2}{t}\right) \\
&= u\left(-\frac{y^2}{t}\right) \left(h\left(-\frac{y}{y'}\right) w\left(\frac{y'^2}{t}\right) \right) w\left(-\frac{yy'}{t}\right)^{-1} u\left(\frac{y^2}{t}\right) \\
&= u\left(\frac{y^2}{t}\right)^{-1} \left(h\left(-\frac{y}{y'}\right) h\left(\frac{y'^2}{t}\right) h\left(-\frac{yy'}{t}\right)^{-1} \right) u\left(\frac{y^2}{t}\right) = c\left(-\frac{y}{y'}, \frac{y'^2}{t}\right).
\end{aligned}$$

□

Corollary 4.2 together with the existence of finitely many explicit formulas for H in terms of c_{12} described in Proposition 4.3 give us the following conclusion.

Corollary 4.4. *Assume that $c': k^\times \times k^\times \rightarrow A$ is a symplectic Steinberg symbol and that $H: \mathrm{SL}_2(k) \times \mathrm{SL}_2(k) \rightarrow K_2^{\mathrm{sym}}(k)$ is the 2-cocycle defined in Proposition 4.3. Let the 2-cocycle $H_{c'}: \mathrm{SL}_2(k) \times \mathrm{SL}_2(k) \rightarrow A$ be obtained from H and c' as it was described in Corollary 4.2. Then $\mathrm{Im}(H_{c'}) \subseteq \mathrm{Im}(c') - \mathrm{Im}(c')$. In particular, if c' has finite image, so has $H_{c'}$.*

Proposition 4.5. *Suppose that $c': k^\times \times k^\times \rightarrow A$ is a symplectic Steinberg symbol. Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = w_{12}(-1)$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = u_{12}(1)$, and let $\langle S, T \rangle$ be the subgroup of $\mathrm{SL}_2(k)$ generated by S and T .*

- (1) *If the restriction of $H_{c'}$ to $\langle S, T \rangle$ is splitting via a function $f: \langle S, T \rangle \rightarrow A$, then $c'(-1, -1) = -12f(T)$.*
- (2) *Assume that $c'(-1, -1) \notin 12 \cdot A$ or $\mathrm{Hom}((k, +), A) = 0 \wedge c'(-1, -1) \neq 0$. Then $H_{c'}$ on $\mathrm{SL}_2(k)$ is a non-splitting 2-cocycle.*

The proof of (1) below is inspired by the proof of [1, Theorem 1].

Proof. (1) By assumption, for every $A, A' \in \langle S, T \rangle$, $H_{c'}(A, A') = f(A) + f(A') - f(AA')$. Furthermore, as a consequence of (S1) and (S2), we have that for all $x \in k^\times$, $c'(1, x) = c'(x, 1) = 0$ (put $z = y = 1$ in (S1); then put $x = 1$ and $z = y^{-1}$ in (S1)). Denote $I = h_{12}(1)$. Then $S^2 = (ST)^3 = -I$ and $ST = w_{12}(-1)u_{12}(1)$, $(ST)^2 = u_{12}(-1)w_{12}(-1)$. Therefore, by Proposition 4.3, we get

$$\begin{aligned}
0 &= c'(1, 1) = H_{c'}(I, I) = f(I), \\
c'(-1, -1) &= H_{c'}(-I, -I) = 2f(-I), \\
0 &= c'(1, 1) = H_{c'}(S, T) = f(S) + f(T) - f(ST), \\
0 &= -c'(-1, 1) = H_{c'}(S, S) = 2f(S) - f(-I), \\
0 &= c'(-1, 1) - c'(-1, 1) = H_{c'}(ST, ST) = 2f(ST) - f((ST)^2), \\
0 &= -c'(-1, 1) = H_{c'}(ST, ((ST)^2)) = f(ST) + f((ST)^2) - f(-I).
\end{aligned}$$

Hence, $2f(ST) = 2f(T) + f(-I)$ and $3f(ST) = f(-I)$, so $c'(-1, -1) = 2f(-I) = -12f(T)$.

(2) Suppose for a contradiction that $H_{c'}$ is splitting via a function $f: \mathrm{SL}_2(k) \rightarrow A$. Then, the restriction of $H_{c'}$ to $\langle S, T \rangle$ is also splitting via f . Thus, the case $c'(-1, -1) \notin 12 \cdot A$ follows by (1). To get a contradiction in the other case, consider the subgroup $U := \{u_{12}(x) : x \in k\}$ of $\mathrm{SL}_2(k)$. By Proposition 4.3, f restricted to U is a homomorphism from U to A . Since $U \cong (k, +)$ and $\mathrm{Hom}((k, +), A)$ is trivial, the function f is identically zero on U . In particular, $f(T) = 0$. Hence, by applying (1), we get that $c'(-1, -1) = 0$, a contradiction. \square

Corollary 4.6. *Suppose that $c': k^\times \times k^\times \rightarrow A$ is a symplectic Steinberg symbol and $\mathrm{char}(k) = 0$.*

(1) *Assume that $c'(-1, -1) \notin 12 \cdot A$. Take as G an arbitrary group with*

$$\mathrm{SL}_2(\mathbb{Z}) \leq G \leq \mathrm{SL}_2(k).$$

Then the restriction of $H_{c'}$ to G is a non-splitting 2-cocycle.

(2) *Assume that $c'(-1, -1) \notin 12 \cdot A$ or $\mathrm{Hom}((\mathbb{Q}, +), A) = 0 \wedge c'(-1, -1) \neq 0$. Take as G an arbitrary group with*

$$\mathrm{SL}_2(\mathbb{Q}) \leq G \leq \mathrm{SL}_2(k).$$

Assume that $\mathrm{SL}_2(\mathbb{Q})$, G , A and the 2-cocycle $H_{c'|_{G \times G}}$ are B -definable in some first order structure \mathcal{G} . Let $\mathcal{G}^ \succ \mathcal{G}$ be a monster model. Then the 2-cocycle $H_{c'|_{G^{*00}_B \times G^{*00}_B}}: G^{*00}_B \times G^{*00}_B \rightarrow A^*$ is non-splitting.*

Proof. (1) It is enough to use Proposition 4.5(1) and a well-known fact that $\mathrm{SL}_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in $\mathrm{SL}_2(k)$ (because $\mathrm{char}(k) = 0$).

(2) Suppose for a contradiction that $H_{c'|_{G^{*00}_B \times G^{*00}_B}}$ is splitting. The group $\mathrm{SL}_2(\mathbb{Q})$ is absolutely connected, so $\mathrm{SL}_2(\mathbb{Q}) < \mathrm{SL}_2(\mathbb{Q})^* = \mathrm{SL}_2(\mathbb{Q})^{*00} \leq G^{*00}_B$. Therefore, the restriction of $H_{c'}$ to $\mathrm{SL}_2(\mathbb{Q})$ is splitting, which contradicts Proposition 4.5(2). \square

The second part of the previous corollary cannot be easily generalized to $\mathrm{SL}_2(\mathbb{Z})$, since this group is not absolutely connected. It has many finite index subgroups. Namely, every homomorphism $\varphi_n: \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ induces a homomorphism $\overline{\varphi}_n: \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$, and $\ker(\overline{\varphi}_n)$ has finite index.

Question 4.7. Let c' be the Steinberg symbol on \mathbb{Q} considered in Example 4.9 below. Is there any first order expansion of $\mathrm{SL}_2(\mathbb{Z})$ (in which $\mathrm{SL}_2(\mathbb{Z})$, \mathbb{Z} and $H_{c'}$ are definable) such that the 2-cocycle $H_{c'}$ is non-splitting on $\mathrm{SL}_2(\mathbb{Z})^{*00}_\emptyset$?

We give a criterion for $H_{c'}$ to be strongly non-splitting in the sense of Corollary 2.2(4) and Proposition 3.4.

Corollary 4.8. *Suppose that $c': k^\times \times k^\times \rightarrow A$ is a symplectic Steinberg symbol and $\mathrm{char}(k) = 0$. Assume that $A = \mathbb{Z}$ and $c'(-1, -1) = 1$. If G is any subgroup of $\mathrm{SL}_2(k)$ containing $\mathrm{SL}_2(\mathbb{Q})$, then $H_{c'}$ restricted to G is strongly non-splitting, that is for every proper subgroup $A' \subsetneq A$ the induced 2-cocycle $\overline{H_{c'}}: G \times G \rightarrow A/A'$ is non-splitting.*

Proof. Notice that $\overline{H_{c'}} = H_{\overline{c'}}$, where $\overline{c'}: k^\times \times k^\times \rightarrow A/A'$ is the induced Steinberg symbol. Suppose for a contradiction that $H_{\overline{c'}}$ is splitting on G . Then $H_{\overline{c'}}$ is splitting on $\mathrm{SL}_2(\mathbb{Q})$, which contradicts Proposition 4.5(2). \square

Now, we are ready to give new classes of examples of groups in which the smallest invariant subgroup of bounded index is a proper subgroup of the smallest type-definable subgroup of bounded index.

Example 4.9. Suppose k is an ordered field with the order denoted by $<$ (e.g. $k = \mathbb{Q}$ or k is an arbitrary subfield of \mathbb{R} with the natural order). One can check that the following mapping $c': k^\times \times k^\times \rightarrow \mathbb{Z}$ is a symplectic Steinberg symbol

$$(\text{****}) \quad c'(x, y) = \begin{cases} 1 & \text{if } x < 0 \text{ and } y < 0 \\ 0 & \text{otherwise} \end{cases}.$$

Let $H: \mathrm{SL}_2(k) \times \mathrm{SL}_2(k) \rightarrow K_2^{\mathrm{sym}}(k)$ be the 2-cocycle defined in Proposition 4.3. By $H_{c'}: \mathrm{SL}_2(k) \times \mathrm{SL}_2(k) \rightarrow \mathbb{Z}$ we denote the 2-cocycle obtained from H and c' as it was described in Corollary 4.2.

Let $\mathcal{G} = ((\mathbb{Z}, +), (k, +, \cdot, <))$, $G = \mathrm{SL}_2(k)$ and $A = (\mathbb{Z}, +)$. The groups G and A are \emptyset -definable in \mathcal{G} . Assume the action of G on A is trivial, and let \tilde{G} (a central extension of G by \mathbb{Z}) be defined by means of the 2-cocycle $H_{c'}$. Let $\mathcal{G}^* = ((\mathbb{Z}^*, +), (k^*, +, \cdot, <)) \succ \mathcal{G}$ be a monster model. Put as A_1^* the connected component \mathbb{Z}^{*0} of the pure group $(\mathbb{Z}^*, +)$ (i.e. the intersection of all groups $n\mathbb{Z}^*$, $n \in \mathbb{N} \setminus \{0\}$), and $B = \{-1, 0, 1\} \subseteq \mathbb{Z}$.

Then the assumptions of Corollary 2.2 are satisfied. So, we get $\tilde{G}_{B}^{*00} \neq \tilde{G}_{B}^{*00} = \tilde{G}^*$, where \tilde{G}^* is the interpretation of \tilde{G} in the monster model \mathcal{G}^* . Moreover, the quotient $\tilde{G}_{B}^{*00} / \tilde{G}_{B}^{*00}$ is abelian. In fact, $\tilde{G}_{B}^{*00} = (\mathbb{Z}^{*0} + \mathbb{Z}) \times G^*$, and $\tilde{G}_{B}^{*00} / \tilde{G}_{B}^{*00}$ is isomorphic to $\hat{\mathbb{Z}} / \mathbb{Z}$, where $\hat{\mathbb{Z}}$ is the profinite completion of \mathbb{Z} .

Proof. The fact that the image of $H_{c'}$ is contained in B (which is finite) follows from the definition of c' and Corollary 4.4. The fact that $H_{c'}$ is definable over B in \mathcal{G} follows from the definition of c' and Proposition 4.3. The assumptions (2) and (5) of Corollary 2.2 are clearly satisfied (see the proof of Example 2.3). The assumption (3) (even $G_B^{*00} = G^*$) follows from the absolute connectedness of $\mathrm{SL}_2(k)$. Finally, (1) and (4) are true by Corollary 4.8.

The fact that $\tilde{G}_{B}^{*00} / \tilde{G}_{B}^{*00}$ is abelian follows from the absolute connectedness of $\mathrm{SL}_2(k)$ and Remark 2.5.

To get the desired description of this quotient, one should apply Propositions 2.15 and 2.16 together with Claim 1 in the proof of Theorem 2.1. Namely, by this claim applied to $H := \mathbb{Z}^{*0}$, we have that $\tilde{G}_{B}^{*00} \cap \mathbb{Z}^* = \mathbb{Z}^{*0} + n\mathbb{Z}$ for some $n \in \mathbb{N}$. Using this together with the fact that $\tilde{G}_{B}^{*00} = \tilde{G}^*$ and Proposition 2.15, we get

$$\tilde{G}_{B}^{*00} / \tilde{G}_{B}^{*00} \cong (\mathbb{Z}^* / \mathbb{Z}^{*0}) / ((\mathbb{Z}^{*0} + n\mathbb{Z}) / \mathbb{Z}^{*0}).$$

But Proposition 2.16 tells us that $(\mathbb{Z}^{*0} + n\mathbb{Z}) / \mathbb{Z}^{*0}$ is a dense subgroup of $\mathbb{Z}^* / \mathbb{Z}^{*0}$, which implies that $n = 1$. Thus, once again using Claim 1 in the proof of Theorem 2.1, we conclude that

$$\tilde{G}_{B}^{*00} = (\mathbb{Z}^{*0} + \mathbb{Z}) \times G^* \quad \text{and} \quad \tilde{G}_{B}^{*00} / \tilde{G}_{B}^{*00} \cong \hat{\mathbb{Z}} / \mathbb{Z}.$$

□

Next, we generalize the situation from the above example in the following way.

Example 4.10. Suppose k is an ordered field, and $c': k^\times \times k^\times \rightarrow \mathbb{Z}$ the Steinberg symbol defined in Example 4.9. Let G be an arbitrary group with

$$\mathrm{SL}_2(\mathbb{Q}) \leq G \leq \mathrm{SL}_2(k).$$

Let $A = \mathbb{Z}$ and $B = \{-1, 0, 1\}$. Assume that $\mathrm{SL}_2(\mathbb{Q})$, G , A and the 2-cocycle $H_{c'|G \times G}$ are B -definable in a first order structure \mathcal{G} . For example, \mathcal{G} might be the two-sorted structure with the disjoint sorts $(\mathbb{Z}, +)$ and $(k, +, \cdot, <)$ together with predicates for G and $\mathrm{SL}_2(\mathbb{Q})$. Let $\mathcal{G}^* \succ \mathcal{G}$ be a monster model. Put $A_1^* = \mathbb{Z}^{*0}$, the intersection of all groups $n\mathbb{Z}^*$, $n \in \mathbb{N} \setminus \{0\}$. Then the assumptions of the first part of Theorem 2.1 are satisfied, so $\widetilde{G^*}_B^{000} \neq \widetilde{G^*}_B^{00}$.

If, moreover, $G_B^{*000} = G^*$, then the assumptions of Corollary 2.2 are satisfied too, so $\widetilde{G^*}_B^{00} = \widetilde{G^*}_B$, and $\widetilde{G^*}_B^{00}/\widetilde{G^*}_B^{000}$ is abelian.

Proof. The assumption (ii) of Theorem 2.1 is clearly satisfied, whereas the assumption (i) follows from Corollary 4.6(2). For the moreover part, it is enough to use Corollary 4.8 and Remark 2.5. \square

Starting from Examples 4.9 or 4.10, the results of Section 3 yield more general classes of examples, some of which are briefly discussed below. In order to avoid a clash of notation (in Section 3, H was a group, whereas in this section, H is a 2-cocycle), the 2-cocycles H and $H_{c'}$ considered in Example 4.9 will be denoted by h and $h_{c'}$, respectively.

Example 4.11. Consider the situation from Example 4.9 (as it is mentioned above, instead of H and $H_{c'}$ we write h and $h_{c'}$). Let

$$1 \hookrightarrow \ker(f) \hookrightarrow H \xrightarrow{f} G \twoheadrightarrow 1$$

be an extension of G by $\ker(f)$. Let \mathcal{H} be any expansion of \mathcal{G} in which H and f are \emptyset -definable (e.g. \mathcal{H} is the expansion of \mathcal{G} by the new sort H together with the function f), and let $\mathcal{H}^* \succ \mathcal{H}$ be a monster model. Assume additionally that $\mathrm{Hom}(\ker(f^*), \mathbb{Z})$ is trivial (where f^* is the interpretation of f in \mathcal{H}^*). Put $h' := h_{c'} \circ (f, f): H \times H \rightarrow \mathbb{Z}$ a 2-cocycle definable in \mathcal{H} over B . Let \widetilde{H} be the extension of H by \mathbb{Z} corresponding to h' . Then $h'_{|H^{*00}_B \times H^{*00}_B}: H^{*00}_B \times H^{*00}_B \rightarrow \mathbb{Z}$ is non-splitting, and $\widetilde{H}^{*000}_B \neq \widetilde{H}^{*00}_B$.

To see a concrete example arising in this way, take as H the extension of $\mathrm{SL}_2(k)$ (where k is an ordered field) by the divisible hull $\mathbb{Q} \otimes_{\mathbb{Z}} K_2^{\mathrm{sym}}(k)$ of $K_2^{\mathrm{sym}}(k)$ corresponding to the 2-cocycle h , and as $f: H \rightarrow \mathrm{SL}_2(k)$ the projection on the second coordinate. More generally, start from any extension of $\mathrm{SL}_2(k)$ by an abelian group C and take as H the group obtained from this extension by replacing C by its divisible hull.

Another family of examples arising in this way is formed by the groups of the form \widetilde{H} for H ranging over all finite extension of $\mathrm{SL}_2(k)$ (because then $\mathrm{Hom}(\ker(f^*), \mathbb{Z})$ is clearly trivial).

Proof. Using Corollary 3.2, the conclusion follows from the observations that $\mathrm{SL}_2(k)$ is absolutely connected and that the assumptions of Theorem 2.1 are satisfied in Example 4.9. \square

Example 4.12. Consider the situation from Example 4.10. Let H be the product of groups $K \times G$ for an arbitrary group K . We define \mathcal{H} as the expansion of \mathcal{G} obtained by adding a new sort, consisting of the pure group structure on H , and the projection

$f: H \rightarrow \mathrm{SL}_2(k)$ on the second coordinate. Let $\mathcal{H}^* \succ \mathcal{H}$ be a monster model and f^* the interpretation of f in it. Put $h' := h_{\mathcal{C}'} \circ (f, f): H \times H \rightarrow \mathbb{Z}$ a 2-cocycle definable in \mathcal{H} over B . Let \widetilde{H} be the extension of H by \mathbb{Z} corresponding to h' . Then $h'_{|H^{*00}_B \times H^{*00}_B}: H^{*00}_B \times H^{*00}_B \rightarrow \mathbb{Z}$ is non-splitting, and $\widetilde{H}^{*00}_B \neq \widetilde{H}^{*00}_B$.

Proof. Using Remark 3.3, the conclusion follows from the fact that the assumptions of the first part of Theorem 2.1 are satisfied in Example 4.10. \square

Although in Example 4.9 the assumptions of both parts of Theorem 2.1 (even of Corollary 2.2) are satisfied (and, in consequence, $\widetilde{G}^{*00}_B \neq \widetilde{G}^{*00}_B = \widetilde{G}^*$), in Example 4.11, we only concluded that the assumptions of the first part of Theorem 2.1 hold, and, in consequence, $\widetilde{H}^{*00}_B \neq \widetilde{H}^{*00}_B$. It is rather clear that in general, we can not expect that $\widetilde{H}^{*00}_B = \widetilde{H}^*$. For example, take $H := \mathbb{Z}/n\mathbb{Z} \times \mathrm{SL}_2(k)$, $f: H \rightarrow \mathrm{SL}_2(k)$ the projection on the second coordinate, and $\mathcal{H} = \mathcal{G} = ((\mathbb{Z}, +), (k, +, \cdot, <))$ where H is definable in the obvious way. Then $\{0 + n\mathbb{Z}\} \times \mathrm{SL}_2(k)$ is a finite index, B -definable (in \mathcal{H}) subgroup of H^* , so $H^{*00}_B \neq H^*$. Therefore, $\widetilde{H}^{*00}_B \neq \widetilde{H}^*$.

However, from Proposition 3.4, we know that if $\ker(f)$ is a finite abelian group and a 2-cocycle h_f corresponding to the extension

$$1 \hookrightarrow \ker(f) \hookrightarrow H \xrightarrow{f} \mathrm{SL}_2(k) \twoheadrightarrow 1$$

is strongly non-splitting, then H is absolutely connected (since $\mathrm{SL}_2(k)$ is such). The next example shows that even in such a situation, it may happen that $\widetilde{H}^{*00}_B \neq \widetilde{H}^*$.

Example 4.13. Consider the situation from Example 4.9. In particular, $G = \mathrm{SL}_2(k)$, $A = \mathbb{Z}$ and the extension \widetilde{G} is given by the 2-cocycle $h_{\mathcal{C}'}: G \times G \rightarrow A$. Let $h_f: G \times G \rightarrow \mathbb{Z}/n\mathbb{Z}$ (for some $n \in \mathbb{N} \setminus \{0\}$) be the 2-cocycle induced by $h_{\mathcal{C}'}$, H be the corresponding extension of G by $\mathbb{Z}/n\mathbb{Z}$, and $f: H \rightarrow G$ be the projection on the second coordinate. Recall that the 2-cocycle $h': H \times H \rightarrow \mathbb{Z}$ is defined as $h_{\mathcal{C}'} \circ (f, f)$, and \widetilde{H} is the corresponding extension of H by \mathbb{Z} . Define $\mathcal{H} = \mathcal{G} = ((\mathbb{Z}, +), (k, +, \cdot, <))$. Then everything is B -interpretable in \mathcal{G} . As usual, $\mathcal{H}^* = \mathcal{G}^* \succ \mathcal{G}$ is a monster model.

Then H is absolutely connected (so $H^{*00}_B = H^*$), but $\widetilde{H}^{*00}_B \neq \widetilde{H}^{*00}_B \neq \widetilde{H}^*$.

Proof. Since by Example 4.9, $h_{\mathcal{C}'}$ is strongly non-splitting, so is h_f . Thus, the absolute connectedness of H follows from Proposition 3.4 and the fact that $\mathrm{SL}_2(k)$ is absolutely connected.

The fact that $\widetilde{H}^{*00}_B \neq \widetilde{H}^{*00}_B$ was proved in Example 4.11. It remains to show that $\widetilde{H}^{*00}_B \neq \widetilde{H}^*$.

Of course, $\widetilde{H}/n\mathbb{Z}$ is B -definably isomorphic with the extension of H by $\mathbb{Z}/n\mathbb{Z}$ corresponding to the 2-cocycle $\overline{h'}: H \times H \rightarrow \mathbb{Z}/n\mathbb{Z}$ induced by h' ; denote this extension by K .

Let $i_n: \widetilde{G} \rightarrow H$ be a B -definable homomorphism defined by $i_n(a, x) = (a + n\mathbb{Z}, x)$. Then $\overline{h'} = i_n \circ h_{\mathcal{C}'} \circ (f, f)$ (after the appropriate identifications).

Define a section $z: G \rightarrow \widetilde{G}$ by $z(x) = (0, x)$. Then $z_n := i_n \circ z: G \rightarrow H$ is a section of f .

We have

$$\begin{aligned}
\overline{h'}(x, y) &= (i_n \circ h_{c'} \circ (f, f))(x, y) = i_n(z(f(x))z(f(y))z(f(xy))^{-1}) \\
&= i_n(z(f(x)))i_n(z(f(y)))i_n(z(f(xy))) = z_n(f(x))z_n(f(y))z_n(f(xy))^{-1} \\
&= (z_n(f(x))x^{-1})x(z_n(f(y))y^{-1})x^{-1}(z_n(f(xy))(xy)^{-1})^{-1} \\
&= F(x) + F(y) - F(xy),
\end{aligned}$$

where $F: H \rightarrow \mathbb{Z}/n\mathbb{Z}$ is defined by $F(x) = z_n(f(x))x^{-1}$. This means that $\overline{h'}$ is splitting via the B -definable function F . Therefore, K is B -definably isomorphic with the product $\mathbb{Z}/n\mathbb{Z} \times H$. Hence, K^* is B -definably isomorphic with $\mathbb{Z}^*/n\mathbb{Z}^* \times H^*$. Since clearly $(\mathbb{Z}^*/n\mathbb{Z}^* \times H^*)_B^{00} \leq \{0 + n\mathbb{Z}^*\} \times H^* \neq \mathbb{Z}^*/n\mathbb{Z}^* \times H^*$, we get that $K_B^{*00} \neq K^*$, which implies $(\widetilde{H}^*/n\mathbb{Z}^*)_B^{00} \neq \widetilde{H}^*/n\mathbb{Z}^*$, and so $\widetilde{H}_B^{*00} \neq \widetilde{H}^*$. \square

Question 2.6 asks if one can find an example with $\widetilde{G}_B^{*00}/\widetilde{G}_B^{*000}$ non-abelian. In Examples 4.9 and 4.13, this quotient is always abelian. But Examples 4.10, 4.11 and 4.12 leave more freedom, and it is not clear if one can find a concrete realization of one of these examples answering Question 2.6 in the affirmative.

The following question is also interesting.

Question 4.14. Does there exist an abelian group G (defined over \emptyset in a monster model) for which $G_\emptyset^{000} \neq G_\emptyset^{00}$. Can one find such a group using Theorem 2.1?

We finish with a discussion on Steinberg symbols. Note that a given field may have many different orders. Each of them gives rise to some symplectic Steinberg symbol, yielding various classes of examples of groups described above.

Recall that a field k is an ordered field (with respect to some order) if and only if it is formally real (i.e. -1 is not a sum of squares in k). Proposition 4.5 and Example 4.9 lead to the following question:

Can a non-formally real field k have a non-trivial symplectic Steinberg symbol $c: k^\times \times k^\times \rightarrow \mathbb{Z}^n$ with finite image?

We answer this question in the negative for fields of characteristic different from 2.

For a field k , by $S(k)$ we denote the set of sums of squares $\{\sum_{i=1}^n a_i^2 : a_i \in k^\times\}$ of k .

Proposition 4.15. *Suppose k is a field, $\text{char}(k) \neq 2$ and $c: k^\times \times k^\times \rightarrow A$ is a symplectic Steinberg symbol with finite image, where A is a torsion free abelian group. Then for every $0 \neq s \in S(k)$ and $t \in k^\times$*

$$c(s, t) = c(t, s) = 0.$$

Proof. We use the relations (S1), (S2) and (S3) as well as the following formulas, which are consequences of (S1) – (S3) (see [9, Proposition 5.7, p. 28]): for $x, y \in k^\times$

- (1) $c(x, y) = c(y^{-1}, x)$,
- (2) $c(x, y) = c(x, -xy)$,
- (3) the mapping $t \mapsto c(x, t^2)$ is a homomorphism from k^\times to A .

We will prove that $c(s, t) = c(t, s) = 0$ whenever $s = \sum_{i=1}^n a_i^2$ for $a_i \in k^\times$ and $t \in k^\times$. We will do it by induction on n .

Case $n = 1$. Since the image of c is finite and A is torsion free, (3) implies that the mapping $t \mapsto c(x, t^2)$ is trivial for an arbitrary $x \in k^\times$. Thus, by (1), $c(s, t) = c(t, s) = 0$.

Before proving the inductive step, we prove the following relations: for $x, y, z \in k^\times$

$$(4) \quad c(xy^2, z) = c(x, zy^2) = c(x, z),$$

$$(5) \ c(x, y) = c(y, x),$$

$$(6) \ c(x, -1) = c(x, y) + c(x, -y).$$

(4) Using (S1) and the case $n = 1$, we have $c(x, zy^2) = c(x, z) + c(xz, y^2) - c(z, y^2) = c(x, z)$. The proof of $c(xy^2, z) = c(x, z)$ is similar.

(5) By (1) and (4), $c(x, y) = c(y^{-1}, x) = c(y^{-1}y^2, x) = c(y, x)$.

(6) By (S1), we have $c(x, -1) = c(x, y(-y^{-1})) = c(x, y) + c(xy, -y^{-1}) - c(y, -y^{-1})$. Moreover, by (2), $c(y, -y^{-1}) = c(y, 1) = 0$ and by (5), (2) and (4), $c(xy, -y^{-1}) = c(-y^{-1}, xy) = c(-y^{-1}, x) = c(-y, x) = c(x, -y)$. Hence we get (6).

Inductive step $n \rightarrow n + 1$. Let $s = \sum_{i=1}^{n+1} a_i^2$. First, we prove that $c(s, -t) = c(s, t)$. We may assume that $s \neq a_1^2$, because otherwise we are done by the case $n = 1$. Then $1 + \sum_{i=2}^{n+1} \left(\frac{a_i}{a_1}\right)^2 \neq 1$, hence we have

$$\begin{aligned} c(s, t) &\stackrel{(4)}{=} c\left(1 + \sum_{i=2}^{n+1} \left(\frac{a_i}{a_1}\right)^2, t\right) \stackrel{(S3)}{=} c\left(1 + \sum_{i=2}^{n+1} \left(\frac{a_i}{a_1}\right)^2, -\left(\sum_{i=2}^{n+1} \left(\frac{a_i}{a_1}\right)^2\right)t\right) \\ &\stackrel{(4)}{=} c\left(s, \left(\sum_{i=2}^{n+1} \left(\frac{a_i}{a_1}\right)^2\right)(-t)\right). \end{aligned}$$

By (S1) and the induction assumption,

$$\begin{aligned} c\left(s, (-t)\left(\sum_{i=2}^{n+1} \left(\frac{a_i}{a_1}\right)^2\right)\right) &= c(s, -t) + c\left(-st, \sum_{i=2}^{n+1} \left(\frac{a_i}{a_1}\right)^2\right) - c\left(-t, \sum_{i=2}^{n+1} \left(\frac{a_i}{a_1}\right)^2\right) \\ &= c(s, -t). \end{aligned}$$

If $t = -1$, then $c(s, -1) = c(s, 1) = 0$ by the case $n = 1$. Hence, by (6), $0 = c(s, -1) = c(s, t) + c(s, -t) = 2c(s, t)$, so $c(s, t) = 0$ and $c(t, s) = 0$ by (5). \square

Corollary 4.16. *If k is a non-formally real field and $\text{char}(k) \neq 2$, then every symplectic Steinberg symbol $c: k^\times \times k^\times \rightarrow A$ with finite image, where A is a torsion free abelian group, is trivial.*

Proof. By assumption, $-1 = a_1^2 + \dots + a_n^2$ is a sum of squares in k . Then every $x \in k^\times$ can be written as a sum of squares $x = \left(\frac{x+1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^2 = \left(\frac{x+1}{2}\right)^2 + \left(a_1 \frac{x-1}{2}\right)^2 + \dots + \left(a_n \frac{x-1}{2}\right)^2$. Hence, by the previous proposition, c is trivial. \square

Question 4.17. Does there exist an infinite field of characteristic 2 possessing a non-trivial symplectic Steinberg symbol with finite image contained in a torsion free abelian group?

There exists a more general theory of central extensions of Chevalley groups (see [9]) via symbols. However, when G is not SL_2 and not of symplectic type, then every symbol $c': k^\times \times k^\times \rightarrow A$ is bimultiplicative, so $c'(xy, z) = c'(x, z) + c'(y, z)$. Therefore, if $A = \mathbb{Z}^n$, then every non-trivial c' has infinite image, so our approach cannot be applied.

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